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Strategy-proof assignment of multiple resources

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Abstract

We examine the strategy-proof allocation of multiple resources; an application is the assignment of packages of tasks, workloads, and compensations among the members of an organization. In the domain of multidimensional single-peaked preferences, we find that any allocation mechanism obtained by maximizing a separably concave function over a polyhedral extension of the set of Pareto-efficient allocations is strategy-proof. Moreover, these are the only strategy-proof, unanimous, consistent, and resource-monotonic mechanisms. These mechanisms generalize the parametric rationing mechanisms (Young, 1987), some of which date back to the Babylonian Talmud.

JEL classification: D70; D63; D47; D61

Keywords: Strategy-proofness; Consistency; Resource-monotonicity

1. Introduction

This paper introduces incentive compatible mechanisms to allocate multiple resources. Applications include the assignment of bundles of tasks, workloads, support personnel, and compensations among a research staff or among an academic department's faculty. In these allocation problems cash transfers are constrained or impossible, resources are not necessarily disposable,

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and preferences cannot be assumed to be monotone. This paper studies the case where preferences over assignments are "multidimensional single-peaked": an agent has an ideal amount of each resource; increases in the amount of a single resource leaving her below the ideal for that resource make her better off, increases beyond it make her worse off.

As in most economic design problems, the relevant information to evaluate the welfare impact of choosing a mechanism, the preferences of the agents involved, is privately held. Successful real-life mechanisms overcome this difficulty and the resulting incentives for manipulation by making truthful preference revelation a dominant strategy. These mechanisms are known as *strategy-proof* and examples include the matching mechanisms in school choice (Abdulkadiroğlu and Sönmez, 2003; Pathak and Sönmez, 2008; Abdulkadiroğlu et al., 2009), kidney exchange (Roth et al., 2004, 2005), and entry level labor markets (as surveyed by Roth, 2002). The focus on dominant strategy incentive compatibility is due to its minimal assumptions about agents' knowledge and behavior. Since reporting preferences truthfully is a dominant strategy, equilibrium behavior does not depend on beliefs, common knowledge of rationality and the information structure, etc. This gives a predictive power and a robustness that are important for practical mechanism design (Wilson, 1987; Bergemann and Morris, 2005).

Unfortunately, in the resource allocation problems studied here, sequential dictatorship is essentially the only strategy-proof and efficient mechanism.[1] This mechanism is neither individually rational nor equitable. Often these distributional objectives will override efficiency and thus exclude this mechanism. In other words, the mechanism designer faces a tradeoff between efficiency and any other objective she may want to implement. This paper describes the class of strategy-proof mechanisms that avoid a number of drawbacks once efficiency is relaxed.

First, we exclude the most inefficient mechanisms. Every mechanism in the class is *unanimous*: if an allocation yielding each agent her ideal assignment is feasible, then the mechanism delivers this allocation. Though sequential dictatorship is the only efficient mechanism in the class, strongly egalitarian mechanisms are also members.

Second, we exclude mechanisms that recommend allocations contradicting each other. A mechanism is *consistent* if its recommendations in problems involving different groups of agents and resources are coherent.[2]

Third, we exclude mechanisms not responding well to changes in the availability of resources. A mechanism is *resource-monotonic* if all agents are made at least as well off in response to certain changes in the availability of resources that can unambiguously make everyone better off. This embodies a basic solidarity notion.[3]

Every strategy-proof, unanimous, consistent, and resource-monotonic mechanism is specified by a list of strictly concave functions (Theorem 1). These functions determine how heavily an agent's welfare is weighed against another's. According to the scarcity of resources, a function is drawn from this list for each agent and each resource. The sum of these functions is then maximized subject to efficiency constraints. The unique maximizer is the allocation recommended by

[1] Sequential dictatorship is the mechanism whereby agents are arranged sequentially, and resources are allocated accordingly. The first agent in the sequence is assigned her best possible bundle. Conditional on this, the second agent is assigned her best possible bundle, and so forth.

[2] Consistency is one of the most thoroughly studied principles in resource allocation. See Thomson (2011a) for an overview. Balinski (2005) and Thomson (2012) discuss the normative content of consistency which Balinski calls "coherence".

[3] See Thomson (2011b) for an overview of solidarity properties in economic environments.

the mechanism. We call the mechanisms defined in this way *separably concave*. Appropriately specifying the list of strictly concave functions defines mechanisms satisfying additional design objectives: when resources are privately owned, so that each agent starts off with an endowment of the resources, individual rationality with respect to these endowments has implications on the functional forms of the functions. Fairness properties like "no-envy" (Foley, 1967) or "fair net trades" (Schmeidler and Vind, 1972) can also be achieved by appropriately specifying the list of functions (see Section 6).

The rest of this paper is organized as follows. Section 2 overviews the most relevant literature. Section 3 introduces the model. Section 4 introduces the strategic and normative properties of mechanisms. Section 5 introduces the separably concave mechanisms and contains the main results. Section 6 illustrates the flexibility these mechanisms have to accommodate additional design criteria. All proofs are collected in Appendix A.

2. Related literature

Perhaps the simplest resource allocation problem within our framework is an Edgeworth box economy. Already here, Hurwicz (1972) established that no individually rational allocation mechanism is strategy-proof and efficient. In fact, in a two-agent classical collective endowment economy, a strategy-proof and efficient mechanism is dictatorial (Zhou, 1991; Schummer, 1997; Goswami et al., 2014).[4]

These impossibility results depend critically on the multidimensionality of assignments. If a single divisible resource is to be allocated among agents with single-peaked preferences, there is a strategy-proof and efficient mechanism satisfying various equity properties, the "uniform rule" (Sprumont, 1991). Moreover, extensive classes of strategy-proof and efficient mechanisms satisfying other desirable properties are known (Barberà et al., 1997; Moulin, 1999; Massó and Neme, 2007). These properties include consistency (Thomson, 1994a; Dagan, 1996) and various solidarity notions (Thomson, 1994b, 1995, 1997).

Most relevantly, in the allocation of a single resource among agents with single-peaked preferences, Moulin (1999) characterized the class of strategy-proof, efficient, and consistent mechanisms satisfying a *physical* resource-monotonicity property. These mechanisms are described by means of "fixed paths" along which the resource is distributed. In the special case of our model where a single resource is to be allocated, our model coincides with Moulin's, and the strategy-proof, unanimous, and consistent mechanisms satisfying our *welfare*-based resource-monotonicity property (see Section 4) are in fact efficient and satisfy Moulin's physical resource-monotonicity property (see Section 6). This has three consequences: it gives an intuitive description of Moulin's mechanisms in terms of the maximization of a separably concave function akin to a social welfare function, it tightens Moulin's characterization since unanimity is weaker than efficiency (Theorem 2), and it replaces Moulin's physical resource-monotonicity property by a more appealing welfare-based property (Theorem 1).

Moving beyond the allocation of a single resource, the natural multidimensional extension of Moulin's resource-monotonicity condition ("physical resource-monotonicity" in Section 5) is even less compelling than our welfare-based resource-monotonicity property. The physical condition requires that, upon an increase in the availability of one of the resources, no agent's

[4] Essentially, the conclusions are as grim when more than two agents are involved. See Goswami et al. (2014), Serizawa (2002), and the references therein.

assignment of *any* resource decreases. However, the arrival of new resources could potentially be used to compensate agents for giving up parts of their assignments.

Finally, the separably concave mechanisms contribute to a recent literature that extends Sprumont's uniform rule to allocation problems involving multiple resources. Preferences here are also assumed to be multidimensional single-peaked.[5] The extension of the uniform rule proposed by this literature is the only strategy-proof mechanism satisfying a weak efficiency notion and no-envy (Amóros, 2002; Adachi, 2010). Weakening efficiency to unanimity and specifying that agents with the same preferences receive welfare-equivalent assignments essentially singles out this extension of the uniform rule among all strategy-proof mechanisms (Morimoto et al., 2013). In fact, this extension of the uniform rule is a mechanism in our proposed class (Section 6). A systematic study of the joint consequences of strategy-proofness, a weak efficiency notion, and no-envy in broader preference domains is available (Cho and Thomson, 2012); these results establish that the domain of multidimensional single-peaked preferences is "maximal" for the existence of non-trivial strategy-proof and envy-free mechanisms.

3. Model

Agents drawn from a finite set A are assigned bundles consisting of amounts of one or more resources. Let $\mathcal{N}$ denote the subsets of A. The finite set K indexes the different kinds of resources that may be available.

Agents may vary in their capacities to receive the resources. The maximum capacity of agent $i \in A$ to receive resource $k \in K$ is denoted by c_i^k. Thus, assignments lie in $X_i \equiv \{x_i \in \mathbb{R}_+^K :$ for each $k \in K,\ x_i^k \leq c_i^k\}$ where x_i^k denotes the kth coordinate of x_i. We refer to X_i as the **assignment space** of agent i.

Agents are equipped with preferences over their assignment spaces. For each agent, $i \in A$, a typical preference relation is denoted by R_i. As usual, P_i denotes the asymmetric part of R_i. The preference relation R_i is **multidimensional single-peaked** if it possesses a unique maximizer or peak p_i in X_i and, for each pair of distinct assignments x_i and y_i in X_i, $x_i\ P_i\ y_i$ if

$$\text{for each } k \in K, \text{ either } p_i^k \geq x_i^k \geq y_i^k \text{ or } p_i^k \leq x_i^k \leq y_i^k.$$

Let $p(R_i)$ denote the maximizer of R_i over X_i and let $p^k(R_i)$ denote its kth coordinate. Let $\mathcal{R}_i$ denote the class of multidimensional single-peaked preferences defined over X_i. For each group of agents $N \in \mathcal{N}$, let $\mathcal{R}^N$ denote the class of profiles $R \equiv (R_i)_{i \in N}$ such that, for each $i \in N$, $R_i \in \mathcal{R}_i$. Let $p(R)$ denote the profile $(p(R_i))_{i \in N}$.

The range of amounts that can be distributed among a group of agents $N \in \mathcal{N}$ is $M(N) \equiv \{m \in \mathbb{R}_+^K :$ for each $k \in K,\ m^k \leq \sum_{i \in N} c_i^k\}$ where m^k denotes the kth coordinate of m. For each $m \in M(N)$, a **feasible allocation** specifies assignments for each agent in N such that all resources are fully allocated, i.e., the sum of the assignments adds up to m. Let $Z(N, m)$ denote the set of feasible allocations of m among N.

An **economy** involving the agents in $N \in \mathcal{N}$ is the pair (R, m) consisting of the preference profile $R \in \mathcal{R}^N$ and the resource profile $m \in M(N)$. Let $\mathcal{E}^N$ denote the collection of economies involving N.

[5] Barberà et al. (1993) studied social choice in the domain of multidimensional single-peaked domain, only considering strict preferences. We do not exclude indifferences.

4. Allocation mechanisms and their properties

A **mechanism** is a mapping φ that recommends, for each economy (R, m), a unique feasible allocation denoted by $\varphi(R, m)$. We now introduce the strategic and normative properties of mechanisms. Unless otherwise specified, we state definitions with respect to a generic group of agents $N \in \mathcal{N}$ and a generic mechanism φ.

We start by recalling the classical efficiency notion. An allocation $x \in Z(N, m)$ is Pareto efficient at $(R, m) \in \mathcal{E}^N$ if there is no $y \in Z(N, m)$ such that, for each $i \in N$, $y_i \; R_i \; x_i$ and, for at least one $i \in N$, $y_i \; P_i \; x_i$. For each $(R, m) \in \mathcal{E}^N$, let $P(R, m)$ denote the set of efficient allocations.

Efficiency: For each $(R, m) \in \mathcal{E}^N$, $\varphi(R, m) \in P(R, m)$.

A minimal efficiency requirement is that the unanimously best allocation is chosen whenever feasible.

Unanimity: For each $(R, m) \in \mathcal{E}^N$ such that $p(R) \in Z(N, m)$, $\varphi(R, m) = p(R)$.

We turn to strategic issues. As discussed before, strategy-proofness is the most compelling incentive compatibility criterion.

Strategy-proofness: For each $(R, m) \in \mathcal{E}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $\varphi_i(R, m) \; R_i \; \varphi_i(R'_i, R_{-i}, m)$.

Beyond its implementation appeal, *strategy-proofness* has been advocated on fairness grounds. If a mechanism is not *strategy-proof*, strategic agents can manipulate at the expense of non-strategic agents (Pathak and Sönmez, 2008).

We move on to consistency, a principle introduced in Nash-bargaining by Harsanyi (1959).[6] Harsanyi argued that if an allocation is viewed as a desirable compromise among a group of agents, then it should not be the case that upon receiving their assignments, two agents pooling their resources will arrive at a different compromise. This idea has been key in the analysis of a wide range of allocation problems.

Consistency: For each pair $N, N' \in \mathcal{N}$ such that $N' \subseteq N$, each $(R, m) \in \mathcal{E}^N$, and each $i \in N'$, $\varphi_i((R_j)_{j \in N'}, \sum_{j \in N'} \varphi_j(R, m)) = \varphi_i(R, m)$.

The next property specifies that "favorable" resource changes do not harm any agent. For example, in classical economies with monotone preferences, resources are always scarce. Thus, when the endowment of resources *increases*, it is natural to specify that all agents should be made at least as well-off; this is known as "resource-monotonicity" (see Thomson, 2011b, for a survey). However, in economies with satiated preferences, resources are not necessarily scarce, and more may be harmful. Thus, we will only require that an increase in a scarce resource makes all agents at least as well-off provided that, after the increase, the resource is still scarce. Conversely, we will require that a decrease in a non-scarce resource makes all agents at least as well-off provided that, after the decrease, the resource is still non-scarce. The following definition will allow us to state this formally.

6 Harsanyi (1977, Page 196) calls the property "multilateral equilibrium".

Definition 1. For each $(R, m) \in \mathcal{E}^N$ and each $\tilde{m} \in M(N)$, $\tilde{m}$ is **between** m and $p(R)$ if, for each $k \in K$, either $m^k \leq \tilde{m}^k \leq \sum_{i \in N} p^k(R_i)$ or $m^k \geq \tilde{m}^k \geq \sum_{i \in N} p^k(R_i)$.

In the above definition, the profile of resources $\tilde{m}$ is closer than m to enabling the ideal allocation where each agent is assigned her ideal assignment or peak. Thus, we require that all agents are at least as well-off under $\tilde{m}$ than under m.

Resource-monotonicity: For each $(R, m) \in \mathcal{E}^N$ and each $\tilde{m} \in M(N)$ between m and $p(R)$, for each $i \in N$, $\varphi_i(R, \tilde{m}) \; R_i \; \varphi_i(R, m)$.

In the special case of our model proposed by Sprumont (1991), where a single resource is to be allocated, *resource-monotonicity* coincides with the monotonicity notion proposed by Sönmez (1994). Thomson (1994b) introduced the weaker "one-sided resource-monotonicity" requiring only that the variation in the availability of the resource affects all agents in the same direction welfare-wise. Under *efficiency*, these monotonicity properties coincide (Ehlers, 2002).

5. Separably concave mechanisms

We offer a full description of the class of *strategy-proof*, *unanimous*, *consistent* and *resource-monotonic* mechanisms. As we will show, all these mechanisms maximize a separably concave function over a polyhedral extension of the set of efficient allocations.

To illustrate the separably concave mechanisms in the simplest setting, consider the problem of allocating a divisible amount of administrative work m^k among a group of agents. Agents $1, \ldots, n$ would rather do as little of the work as possible and each can do at most $c_1^k, \ldots, c_n^k$, respectively. The question of how to allocate m^k among $1, \ldots, n$ has been the subject of a whole strand of research since it was formulated in the context of the adjudication of conflicting claims (O'Neill, 1982).[7] Specific examples and proposed awards can be found in the Babylonian Talmud. However, a systematic procedure or mechanism yielding the awards in these scriptures remained elusive until Aumann and Maschler (1985) succeeded in providing one. Young (1987) then observed that the recommendations made by this mechanism can be computed as solutions to the following optimization problem:

$$\max \textstyle\sum_{i=1}^n u_i(z_i) \quad \text{subject to } \sum_{i=1}^n z_i = m^k \text{ and } 0 \leq z_i \leq c_i^k,$$

where

$$u_i(z_i) \equiv \begin{cases} \ln z_i & \text{if} \quad 0 \leq z_i \leq \frac{c_i^k}{2}, \\ \ln(c_i^k - z_i) & \text{if} \quad \frac{c_i^k}{2} \leq z_i \leq c_i^k. \end{cases}$$

Note that u_i is concave. In fact, the central mechanisms for this problem can be described as solutions to optimization problems analogous to the one above: each "parametric" mechanism (Young, 1987) can be defined by appropriately choosing the u_i functions.[8] Another central allocation mechanism in the parametric class, "constrained equal awards", is obtained by setting $u_i(z_i) = -z_i^2$.

[7] Claims problems have several interpretations (taxation, bankruptcy, rationing, etc.) and are the most thoroughly studied problems in fair allocation. See Thomson (2003) for a survey.

[8] Young (1987) considers a less general class of strictly concave functions where $u_i = u(\cdot, c_i^k)$. Stovall (2014a, 2014b) studies other mechanisms defined by maximizing a separably concave function.

We have assumed that each agent prefers as small a share of the administrative work as possible. These preferences are monotone and hence single-peaked. Moreover, any division of the administrative workload among the agents is (Pareto) efficient with respect to these preferences. Thus, we could also define the mechanism rationalizing the awards in the Talmud as the solution to a maximization problem over the set of efficient allocations:

$$\max \textstyle\sum_{i=1}^{n} u_i(z_i) \quad \text{subject to } (z_1, \dots, z_n) \in P(R_1, \dots, R_n, m^k),$$

where K is assumed to be the singleton $\{k\}$ and preferences $R_1, \dots, R_n$ are assumed to be monotone.

A somewhat surprising observation is that if we were to drop the assumption that $R_1, \dots, R_n$ are monotone preference relations – and just assume single-peakedness – we could compute the recommendations made by the uniform rule (Sprumont, 1991) by solving

$$\max \textstyle\sum_{i=1}^{n} -z_i^2 \quad \text{subject to } (z_1, \dots, z_n) \in P(R_1, \dots, R_n, m^k). \tag{1}$$

This equivalent definition can be derived from either of the following facts: Firstly, the allocation recommended by the uniform rule can be obtained by choosing the unique allocation that minimizes the variance among all efficient allocations (Schummer and Thomson, 1997). Secondly, the allocation recommended by the uniform rule is Lorenz dominant among all efficient allocations (De Frutos and Massó, 1995).

A central property of the uniform rule is its *strategy-proofness* (Bénassy, 1982; Sprumont, 1991). An insight of this paper is that replacing any of the $-z_i^2$ by *any* strictly concave u_i in (1) defines a *strategy-proof* mechanism. This observation extends to our model with multiple resources.

All of our mechanisms are obtained as solutions to optimization problems similar to the ones above. The main difference, is that when more than one resource is to be allocated, optimization is no longer defined over the efficient set but over a set containing it. In the single resource case, the two sets coincide (see Remark 2 in Appendix A).

Informally, the mechanisms introduced here are specified as follows: for each resource kind k, each agent i is equipped with a pair $(u_i^{xd,k}, u_i^{xs,k})$ of strictly concave and continuous functions over her possible assignments of resource k. The allocation is computed as follows: in situations of **excess demand** (xd) for resource k,[9] the allocation of resource k is chosen so as to maximize $\sum_i u_i^{xd,k}$ while insuring no agent receives more than her preferred consumption of k. In situations of **excess supply** (xs) for resource k,[10] the allocation of resource k is chosen so as to maximize $\sum_i u_i^{xs,k}$ while insuring no agent receives less than her preferred consumption of k.

Formally, let $\mathcal{U}$ denote the profiles $u \equiv \{(u_i^{xd,k}, u_i^{xs,k}) : i \in A, k \in K\}$ where $u_i^{xd,k}, u_i^{xs,k} : [0, c_i^k] \to \mathbb{R}$ are strictly concave and continuous functions. A mechanism φ is **separably concave** if there is a $u \in \mathcal{U}$ such that, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each $k \in K$,

$$\varphi^k(R, m) = \arg\max\{\textstyle\sum_i u_i^{xd,k}(z_i) : \sum_i z_i = m^k, z \in \times_i [0\ , p^k(R_i)]\} \text{ if } \sum_i p^k(R_i) \geq m^k,$$
$$\varphi^k(R, m) = \arg\max\{\textstyle\sum_i u_i^{xs,k}(z_i) : \sum_i z_i = m^k, z \in \times_i [p^k(R_i), c_i^k]\} \text{ if } \sum_i p^k(R_i) \leq m^k,$$

where $i \in N$ and $\varphi^k(R, m)$ specifies the distribution of the amount m^k among the agents in N recommended by φ.

[9] The sum of the preferred consumptions of resource k exceeds the available amount.

[10] The sum of the preferred consumptions of resource k is less than the available amount.

Remark 1. The constraint set in the first optimization problem above is

$$\{z \in \times_{i\in N}[0, p^k(R_i)] : \textstyle\sum_{i\in N} z_i = m^k\}.$$

This is a compact and convex set and $\sum_i u_i^{xd,k}$ is strictly concave and continuous. Thus, the optimization problem has a unique solution. Similarly, the second optimization problem above has a unique solution. Thus, a separably concave mechanism is well defined.

We can now state our main result.

Theorem 1. *The separably concave mechanisms are the only strategy-proof, unanimous, consistent, and resource-monotonic mechanisms.*

All separably concave mechanisms satisfy another monotonicity property: when the supply of each resource increases, no agent's assignment of any resource decreases.

Physical resource-monotonicity: For each $(R, m) \in \mathcal{E}^N$ and each $\tilde{m} \in M(N)$ such that, for each $k \in K$, $m^k \leq \tilde{m}^k$, no agent is assigned less of any resource at $\varphi(R, \tilde{m})$ than at $\varphi(R, m)$.

This intuitive condition underlies the analytical tractability of the separably concave mechanisms yet it is not intended to be normatively compelling since it has no welfare content. In the single resource case, the condition coincides with the monotonicity property proposed by Moulin (1999). Furthermore, in this case and under *efficiency*, *physical resource-monotonicity* coincides with *resource-monotonicity* (Ehlers, 2002). Our final theorem establishes that every *strategy-proof*, *unanimous*, and *consistent* mechanism satisfying either one of our resource-monotonicity properties is in the separably concave family.

Theorem 2. *The separably concave mechanisms are the only strategy-proof, unanimous, consistent, and physically resource-monotonic mechanisms.*

6. Applications

We now illustrate the breadth and flexibility of the separably concave mechanisms to accommodate individual rationality and various distributional objectives. We also derive further implications of our results for the single resource case.

6.1. Equity

A central equity notion in fair allocation is "no-envy" (Foley, 1967). A mechanism φ satisfies no-envy if, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each pair of agents $i, j \in N$, $\varphi_i(R, m)\ R_i\ \varphi_j(R, m)$. That is, the recommended allocations are such that each agent finds her assignment to be at least as desirable as that of any other agent. "Equal treatment of equals", a weaker property, requires that identical agents receive identical assignments. That is, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each pair of agents $i, j \in N$ such that $R_i = R_j$, $\varphi_i(R, m) = \varphi_j(R, m)$. Note that these properties require that agents have the same assignment spaces.

There is a unique separably concave mechanism satisfying either of these properties. The usual definition of this mechanism (Amóros, 2002; Adachi, 2010; Morimoto et al., 2013), the **commodity-wise uniform rule**, U, is as follows: for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, agent

$i \in N$ receives an amount of resource $k \in K$ given by

$$U_i^k(R,m) = \begin{cases} \min\{p^k(R_i), \lambda^k\} & \text{if} \quad \sum_{i\in N} p^k(R_i) \geq m^k, \\ \max\{p^k(R_i), \lambda^k\} & \text{if} \quad \sum_{i\in N} p^k(R_i) \leq m^k, \end{cases}$$

where λ^k is the solution to $\sum_{i\in N} \min\{p^k(R_i), \lambda^k\} = m^k$ if the first case above holds and is the solution to $\sum_{i\in N} \max\{p^k(R_i), \lambda^k\} = m^k$ otherwise.

An alternative definition of the commodity-wise uniform rule, emphasizing its membership in the separably concave class, is as follows: it is the separably concave rule specified by u in $\mathcal{U}$ such that, for each $i \in A$, each $k \in K$, and each $z \in [0, c_i^k]$, $u_i^{xd,k}(z) = u_i^{xs,k}(z) = -z^2$. The arguments establishing this coincidence are the same, repeated resource by resource, as those used to establish the coincidence of the usual definition of the uniform rule (Sprumont, 1991) and that in the optimization problem in (1).

When all agents share the same assignment spaces, the commodity-wise uniform rule is the only *strategy-proof*, *unanimous*, and "non-bossy" (Satterthwaite and Sonnenschein, 1981) mechanism recommending allocations satisfying equal treatment of equals (Morimoto et al., 2013). Non-bossiness requires that an agent is only able to alter another agent's assignment by altering her own. It is straightforward to verify that *consistency* implies non-bossiness (see Lemma 10 in Appendix A). Thus, the commodity-wise uniform rule is singled out, within the separably concave mechanisms, by equal treatment of equals. Since the commodity-wise uniform rule satisfies no-envy (Adachi, 2010), which implies equal treatment of equals, it is also the only separably concave mechanism satisfying no-envy.

6.2. Priorities

In many applications (Abdulkadiroğlu and Sönmez, 2003; Ergin, 2002; Kojima, 2013), agents have different priorities to receive the various resources. It is straightforward to construct separably concave mechanisms respecting such priorities. The simplest such mechanism is a sequential dictatorship discussed in the Introduction.

Suppose that we need to prioritize the agents in A, which we label $\{1, 2, \dots, n\}$, so that agent 1 has the highest priority, agent 2 has the second highest priority, and so forth. This means that, if agent 1 is not being assigned her ideal assignment or peak, there should be no other allocation improving upon her assignment. Conditional on this being achieved, if agent 2 is not being assigned her peak, there should be no alternative allocation improving upon her assignment, and so forth. Note that this mechanism is *efficient*. There is a u in $\mathcal{U}$ such that the corresponding separably concave mechanism implements this priority scheme. For example, for each $i \in \{1, \dots, n\}$, each $k \in K$, and each $z \in [0, c_i^k]$ let

$$u_i^{xd,k}(z) \equiv -iz + \ln(1+z) \text{ and } u_i^{xs,k}(z) \equiv iz + \ln(1+z).$$

Suppose we are in a situation of excess demand for resource k. Then, recall that, under a separably concave mechanism, no agent receives more than her ideal amount of resource k. Conditional on this, a higher priority for resource k means being awarded *more*. The agent with the lowest i is given priority because her marginal return in the optimization problem defining the mechanism is greatest. Agents with higher values of i receive the resource only if the lowest i agent present is awarded her ideal assignment of resource k.

Suppose instead that we are in a situation of excess supply for resource k. Then, recall that, under a separably concave mechanism, no agent receives less than her ideal amount of resource k. Conditional on this, higher priority for resource k means being awarded *less*. The agent with the lowest i is given priority because her marginal return in the optimization problem defining the mechanism is lowest. Agents with higher values of i receive less than their maximum capacity only if the lowest i agent present is awarded her ideal assignment of resource k.

6.3. Individual endowments

To discuss individual rationality and other properties specific to situations with individual endowments, we now account for this data. We specify that each agent $i \in A$ has an endowment of resources ω_i in her assignment space X_i. Then, for each $N \in \mathcal{N}$, we consider the subclass of economies $(R, m) \in \mathcal{E}^N$ such that, $\sum_N \omega_i = m$. These are the economies where the resources to be allocated among the agents in N are precisely the sum of their endowments. For each $N \in \mathcal{N}$, let $\mathcal{E}^N_\omega$ denote this subclass of economies.

A mechanism φ is individually rational if, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N_\omega$, and each $i \in N$, $\varphi_i(R, m)\ R_i\ \omega_i$. That is, we acknowledge an agent's right to receive assignments at least as desirable as her endowment. Many separably concave mechanisms are individually rational. For example, the **generalized commodity-wise uniform rule** is the separably concave mechanism specified by $u \in \mathcal{U}$ such that, for each $i \in A$, each $k \in K$, and each $z \in [0, c^k_i]$, $u^{xd,k}_i(z) = u^{xs,k}_i(z) = -(z - \omega^k_i)^2$.

The notion of fair net trades (Schmeidler and Vind, 1972) extends no-envy to situations with individual endowments. It requires that the way the allocation we recommend *adjusts* over endowments satisfies no-envy. A mechanism φ satisfies fair net trades if, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N_\omega$, and each pair of agents $i, j \in N$, $\varphi_i(R, m)\ R_i\ (\omega_i + \varphi_j(R, m) - \omega_j)$, where $\varphi_j(R, m) - \omega_j$ is the "adjustment" of agent j over her endowment. The requirement is mute when these welfare comparisons are not well defined. The generalized commodity-wise uniform rule satisfies fair net trades.

6.4. Further results for the single resource case

Some properties of the separably concave mechanisms that hold in the single resource case do not hold in general. For example, all separably concave mechanisms are *efficient* and immune to coalition manipulation or "group strategy-proof".

Formally, a mechanism φ is **group strategy-proof** if, for each $N \in \mathcal{N}$, and each $(R, m) \in \mathcal{E}^N$, there are no $N' \subseteq N$ and $R' \in \mathcal{R}^N$ such that, for each $i \in N \setminus N'$, $R_i = R'_i$, and (i) for each $i \in N'$, $\varphi_i(R', m)\ R_i\ \varphi_i(R, m)$ and, (ii) for some $i \in N'$, $\varphi_i(R', m)\ P_i\ \varphi_i(R, m)$.

Proposition 1. *Suppose that K is a singleton. The separably concave mechanisms are group strategy-proof.*

If more than one resource is to be allocated (when the cardinality of K is greater than one), the separably concave mechanisms are not necessarily *group strategy-proof*. The commodity-wise uniform rule is not (Morimoto et al., 2013).

By Theorems 1 and 2, the separably concave mechanisms are the only *strategy-proof*, *unanimous*, and *consistent* mechanisms satisfying either of our resource-monotonicity properties. Moreover, these properties imply efficiency when K is a singleton.

Proposition 2. *Suppose that K is a singleton. A strategy-proof, unanimous, and consistent mechanism is efficient.*

Therefore, we obtain the following:

Corollary 1. *Suppose that K is a singleton.*

(i) *The separably concave mechanisms are the only strategy-proof, efficient, consistent, and resource-monotonic mechanisms.*
(ii) *The separably concave mechanisms are the only strategy-proof, efficient, consistent, and physically resource-monotonic mechanisms.*

For the single resource case, part (ii) of Corollary 1 establishes the coincidence of the separably concave mechanisms with the "fixed-path" mechanisms of Moulin (1999). Apart from representing the class of *strategy-proof*, *efficient*, *consistent*, and *physically resource-monotonic* mechanisms as solutions to optimization problems, our characterization is tighter: Theorem 2 weakens *efficiency* to *unanimity*. Additionally, Theorem 1 replaces *physical resource-monotonicity* by our welfare-based *resource-monotonicity*.

An additional insight from our results is that they bridge a gap with the literature on claims problems discussed in the beginning of Section 5. Claims problems can be formally embedded as special cases of our model, where K is a singleton, preferences are monotone, and the upper capacity constraints are interpreted as claims. As we saw, the separably concave mechanisms subsume the parametric mechanisms of Young (1987), some of which date back to the Babylonian Talmud (Aumann and Maschler, 1985; Young, 1987). Thus, the class of separably concave mechanisms can be viewed as extending and generalizing the parametric mechanisms to the resource allocation problems studied here.

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Appendix A

A.1. Notation and further definitions

The following notation will be used throughout the appendix. Let $\{Y_i\}_{i\in I}$ denote a family of sets Y_i indexed by I. Let $Y^I \equiv \times_{i\in I} Y_i$. For each $y \in Y^I$ and each $J \subseteq I$, y_J denotes the projection of y onto Y^J. If $x, y \in \mathbb{R}^I$, then $x \geq y$ means that, for each $i \in I$, $x_i \geq y_i$. For each

$i \in I$, $\mathbf{e}_i \in \mathbb{R}^I$ denotes the ith standard basis vector, i.e., the vector with a one in the ith coordinate and zeros elsewhere. Given a function $f : \mathbb{R} \to [-\infty, +\infty]$, for each $x \in \mathbb{R}$, $\partial_+ f(x)$ and $\partial_- f(x)$ denote the right hand and left hand derivatives of f at x, respectively.

For each $k \in K$ and each $i \in A$, let $X_i^k \equiv [0, c_i^k]$.

For each $k \in K$ and each $N \in \mathcal{N}$, let $M^k(N) \equiv [0, \sum_{i \in N} c_i^k]$.

For each $k \in K$, each $N \in \mathcal{N}$, and each $R \in \mathcal{R}^N$, let $p^k(R) \equiv (p^k(R_i))_{i \in N}$.

For each $k \in K$, each $N \in \mathcal{N}$, each $r \in \times_{i \in N} X_i^k$, and each $\mu \in M^k(N)$, let

$$S^k(r, \mu) \equiv \{z \in \times_{i \in N} X_i^k : \textstyle\sum_N z_i = \mu, z \le r\}.$$

For each $u \in \mathcal{U}$, each $N \in \mathcal{N}$, and each $(R, m) \in \mathcal{E}^N$, $\phi(R, m; u)$ denotes the allocation recommended by the separably concave mechanism specified by u for the economy (R, m). For each $k \in K$ and each $i \in N$, $\phi_i^k(R, m; u)$ denotes the amount of resource k received by agent i at allocation $\phi(R, m; u)$ and $\phi^k(R, m; u)$ denotes the profile $(\phi_j^k(R, m; u))_{j \in N}$. Likewise, given a mechanism φ, $\varphi_i^k(R, m)$ specifies the amount of resource k received by agent i at allocation $\varphi(R, m)$ and $\varphi^k(R, m)$ denotes the profile $(\varphi_j^k(R, m))_{j \in N}$. For each x in $Z(N, m)$, x_i^k denotes the amount of resource k received by agent i at allocation x and $x^k \equiv (x_j^k)_{j \in N}$.

Non-bossiness: For each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, each $i \in N$, and each $R_i' \in \mathcal{R}_i$, $\varphi_i(R, m) = \varphi_i(R_i', R_{-i}, m)$ implies that $\varphi(R, m) = \varphi(R_i', R_{-i}, m)$.

Converse consistency: For each $N \in \mathcal{N}$ and each $(R, m) \in \mathcal{E}^N$,

$$[x \in Z(N, m) \text{ and, for each } \{i, j\} \subseteq N,\ x_{\{i,j\}} = \varphi(R_{\{i,j\}}, x_i + x_j)] \Rightarrow x = \varphi(R, m).$$

To define "separability" we introduce additional notation. For each $N \in \mathcal{N}$ and each $k \in K$, define the mapping ψ^k, specifying, for each

$$(p^k, m^k) \in [\times_{i \in N} X_i^k] \times M^k(N),$$

a feasible division of m^k among the agents in N,

$$\psi^k(p^k, m^k) \in \{x \in \times_{i \in N} X_i^k : \textstyle\sum_N x_i = m^k\}.$$

Let Ψ denote the class of profiles $\{\psi^k : k \in K\}$ of such mappings.

Separability: There is $\{\psi^k : k \in K\}$ in Ψ such that, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each $k \in K$, $\varphi^k(R, m) = \psi^k(p^k(R), m^k)$.

Given a *separable* mechanism φ, we refer to the corresponding $\{\psi^k : k \in K\}$ in Ψ as the **decomposition** of φ.

Lemma 1. *Let φ denote a physically resource-monotonic and separable mechanism. Then, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, each $\hat{m} \in M(N)$, and each $k \in K$,*

$$\text{if } \hat{m}^k \ge m^k, \text{ then } \varphi^k(R, \hat{m}) \ge \varphi^k(R, m).$$

Proof. Let φ denote a mechanism satisfying the properties in Lemma 1. Let $N \in \mathcal{N}$, $(R, m) \in \mathcal{E}^N$, $\hat{m} \in M(N)$, and $k \in K$. Suppose that $\hat{m}^k \ge m^k$. Let $\tilde{m} \in M(N)$ be such that $\tilde{m}^k = \hat{m}^k$ and $\tilde{m} \ge m$. By *separability*, $\varphi^k(R, \hat{m}) = \varphi^k(R, \tilde{m})$. By *physical resource-monotonicity*, $\varphi^k(R, \tilde{m}) \ge \varphi^k(R, m)$. Thus, $\varphi^k(R, \hat{m}) \ge \varphi^k(R, m)$. □

A.2. Properties of the separably concave mechanisms

Lemma 2. *The separably concave mechanisms are strategy-proof, unanimous, consistent, physically resource-monotonic, separable, resource-monotonic, and conversely consistent.*

Proof. Let $u \equiv \{(u_i^k, v_i^k) : i \in A, k \in K\} \in \mathcal{U}$, $N \in \mathcal{N}$, $(R, m) \in \mathcal{E}^N$, and $x \equiv \phi(R, m; u)$.

Strategy-proofness: Let $i \in N$ and $\tilde{R} \in \mathcal{R}^N$ be such that, for each $j \in N \setminus \{i\}$, $\tilde{R}_j = R_j$, and let $y \equiv \phi(\tilde{R}, m; u)$. We will prove that $x_i \; R_i \; y_i$. By multidimensional single-peakedness, it suffices to prove that, for each $k \in K$, either $y_i^k \leq x_i^k \leq p^k(R_i)$ or $y_i^k \geq x_i^k \geq p^k(R_i)$. Let $k \in K$, $p \equiv p^k(R)$, and $q \equiv p^k(\tilde{R})$.

Case 1. $\sum_N p_j \geq m^k$. Then, by definition, x^k maximizes $\sum_N u_j^k$ over all feasible distributions of m^k at which each agent j receives no more than p_j. If $x_i^k = p_i$ there is nothing to show since, either $y_i^k \geq x_i^k = p_i$ or $y_i^k \leq x_i^k = p_i$, as desired. Suppose instead that $x_i^k < p_i$.

Case 1.1. $\sum_N q_j \leq m^k$. Then, by definition, y^k maximizes $\sum_N v_j^k$ over all feasible distributions of m^k at which each agent j receives at least q_j. Thus, for each $j \in N \setminus \{i\}$, $y_j^k \geq q_j = p_j \geq x_j^k$. Thus, since $\sum_{j \in N} x_j^k = m^k = \sum_{j \in N} y_j^k$, $y_i^k \leq x_i^k \leq p_i$, as desired.

Case 1.2. $\sum_N q_j \geq m^k$. Then, by definition, y^k maximizes $\sum_N u_j^k$ over all feasible distributions of m^k at which each agent j receives no more than q_j. If $q_i \leq x_i^k$, then $y_i^k \leq q_i \leq x_i^k \leq p_i$, as desired. Suppose instead $q_i > x_i^k$. Then, $x^k \in S^k(q, m^k)$ and, by definition, $y^k \in S^k(q, m^k)$.

We now show that $y_i^k = x_i^k$. If $y_i^k < x_i^k$, since, for each $j \in N \setminus \{i\}$, $q_j = p_j$ then $y^k \in S^k(p, m^k)$. But this contradicts x^k being the maximizer of $\sum_N u_j^k$ over $S^k(p, m^k)$. Thus, $y_i^k \geq x_i^k$. If $y_i^k > x_i^k$, there is $j \in N \setminus \{i\}$ such that $y_j^k < x_j^k \leq q_j$. Hence, there is $\varepsilon > 0$ such that $y^k + \varepsilon(\mathbf{e}_j - \mathbf{e}_i) \in S^k(q, m^k)$. Thus, a necessary condition for y^k to maximize $\sum_N u_j^k$ over $S^k(q, m^k)$ is that $\partial_+ u_j^k(y_j^k) \leq \partial_- u_i^k(y_i^k)$, and, since u_i^k and u_j^k are strictly concave, we obtain the first and last inequalities in

$$\partial_- u_j^k(x_j^k) < \partial_+ u_j^k(y_j^k) \leq \partial_- u_i^k(y_i^k) < \partial_+ u_i^k(x_i^k).$$

Thus, $\partial_- u_j^k(x_j^k) < \partial_+ u_i^k(x_i^k)$. Since $x_i^k < p_i$ and $x_j^k > y_j^k \geq 0$, there is $\varepsilon > 0$ such that $x^k + \varepsilon(\mathbf{e}_i - \mathbf{e}_j) \in S^k(p, m^k)$. But this contradicts x^k being the maximizer of $\sum_N u_j^k$ over $S^k(p, m^k)$. Thus, $y_i^k \leq x_i^k$. Thus, in fact, $y_i^k = x_i^k$, as desired.

Case 2. $\sum_N p_j \leq m^k$. By arguments analogous to those used for Case 1, either $y_i^k \geq x_i^k \geq p_i$ or $y_i^k \leq x_i^k \leq p_i$.

Unanimity: Suppose that $p(R) \in Z(N, m)$. Thus, $\sum_{i \in N} p(R_i) = m$. Thus, from the definition of a separably concave mechanism, for each $k \in K$ and each $i \in N$, $x_i^k \leq p^k(R_i)$ and $x_i^k \geq p^k(R_i)$. Thus, for each $k \in K$ and each $i \in N$, $x_i^k = p^k(R_i)$. Thus, $x = p(R)$.

Consistency: Let $N' \in \mathcal{N}$ denote a proper subset of N. By way of contradiction, suppose that $y \equiv \phi(R_{N'}, \sum_{N'} x_j; u) \neq x_{N'}$. Then, there is $k \in K$ such that $x_{N'}^k \neq y^k$. Suppose that $\sum_{i \in N} p^k(R_i) \geq m^k$. By the definition of $\phi(\cdot\,; u)$, for each $i \in N$, $x_i^k \leq p^k(R_i)$ and, for each $i \in N'$, $y_i^k \leq p^k(R_i)$. Then, because the maximization problem defining y^k has a unique solution and because $x_{N'}$ also satisfies the constraints of this maximization problem,

$\sum_{i\in N'} u_i^k(x_i^k) < \sum_{i\in N'} u_i^k(y_i^k)$. Note that, because $\sum_{i\in N'} x_i^k = \sum_{i\in N'} y_i^k$, $z^k \equiv (y^k, x^k_{N\setminus N'}) \leq p^k(R)$ and $\sum_{i\in N} z_i^k = m^k$. Thus, $\sum_{i\in N} u_i^k(x_i^k) < \sum_{i\in N} u_i^k(z_i^k)$. Thus, $x^k \neq \phi^k(R, m; u)$, contradicting $x \equiv \phi(R, m; u)$. A symmetric argument applies if $\sum_{i\in N} p^k(R_i) \leq m^k$.

Physical resource-monotonicity: Let $\hat{m} \in M(N)$ be such that $\hat{m} \geq m$ and $\hat{x} \equiv \phi(R, \hat{m}; u)$. We need to show that $\hat{x} \geq x$. By way of contradiction, suppose that there are $i \in N$ and $k \in K$ such that $x_i^k > \hat{x}_i^k$. By the definition of $\phi(\cdot\,; u)$, if $m^k = \hat{m}^k$, $x^k = \hat{x}^k$. Thus, $\sum_{h\in N} \hat{x}_h^k = \hat{m}^k > m^k = \sum_{h\in N} x_h^k$. Thus, there is $j \in N$ such that $\hat{x}_j^k > x_j^k$. Suppose that $\sum_{h\in N} p^k(R_h) \geq m^k$. Thus, by the definition of $\phi(\cdot\,; u)$, for each $h \in N$, $x_h^k \leq p^k(R_h)$. If $\hat{m}^k \geq \sum_{h\in N} p^k(R_h)$, then, by the definition of $\phi(\cdot\,; u)$, for each $h \in N$, $\hat{x}_h^k \geq p^k(R_h) \geq x_h^k$. Thus, in the case under consideration, $\sum_{h\in N} p^k(R_h) > \hat{m}^k > m^k$. Then, by the definition of $\phi(\cdot\,; u)$, x^k and $\hat{x}^k$ maximize $\sum_{h\in N} u_h^k$ over $S^k(p^k(R), m^k)$ and $S^k(p^k(R), \hat{m}^k)$, respectively. Since $\hat{x}_j^k > x_j^k \geq 0$ and $\hat{x}_i^k < x_i^k \leq p^k(R_i)$, there is $\varepsilon > 0$ such that $\hat{x}^k + \varepsilon(\mathbf{e}_i - \mathbf{e}_j) \in S^k(p^k(R), \hat{m}^k)$. Thus, a necessary condition for $\hat{x}^k$ to maximize $\sum_{h\in N} u_h^k$ over $S^k(p^k(R), \hat{m}^k)$ is that $\partial_+ u_i^k(\hat{x}_i^k) \leq \partial_- u_j^k(\hat{x}_j^k)$. Moreover, since u_i^k and u_j^k are strictly concave, we obtain the first and last inequalities in

$$\partial_+ u_j^k(x_j^k) > \partial_- u_j^k(\hat{x}_j^k) \geq \partial_+ u_i^k(\hat{x}_i^k) > \partial_- u_i^k(x_i^k).$$

Thus, $\partial_+ u_j^k(x_j^k) > \partial_- u_i^k(x_i^k)$. Since $x_i^k > \hat{x}_i^k \geq 0$ and $x_j^k < \hat{x}_j^k \leq p^k(R_j)$, there is $\varepsilon > 0$ such that $x^k + \varepsilon(\mathbf{e}_j - \mathbf{e}_i) \in S^k(p^k(R), m^k)$. Thus, x^k does not maximize $\sum_{h\in N} u_j^k$ over $S^k(p^k(R), m^k)$. This contradiction establishes that, in fact, $\hat{x} \geq x$.

If $\sum_{h\in N} p^k(R_h) \leq m^k$, then arguments analogous to those above again establish that $\hat{x} \geq x$.

Separability: This is clear from the definition of a separably concave mechanism.

Resource-monotonicity: Let $\tilde{m} \in M(N)$ be between m and $p(R)$. Thus, by Lemma 1 and the definition of $\phi(\cdot\,; u)$, for each $k \in K$,

$$p^k(R) \leq \phi^k(R, \tilde{m}; u) \leq \phi^k(R, m; u) \text{ or } p^k(R) \geq \phi^k(R, \tilde{m}; u) \geq \phi^k(R, m; u).$$

By multidimensional single-peakedness, for each $i \in N$, $\phi_i(R, \tilde{m}; u)\ R_i\ x_i$.

Converse consistency: Let $z \in Z(N, m)$ be such that, for each $\{i, j\} \subseteq N$, $z_{\{i,j\}} = \phi(R_i, R_j, z_i + z_j; u)$. Since the separably concave mechanisms are *consistent*, for each $\{i, j\} \subseteq N$, $x_{\{i,j\}} = \phi(R_i, R_j, x_i + x_j; u)$. By way of contradiction, if $\phi(\cdot\,; u)$ is not *conversely consistent*, there is $\{i, j\} \subseteq N$ such that $z_{\{i,j\}} \neq x_{\{i,j\}}$. By the *consistency* of $\phi(\cdot\,; u)$, $z_i + z_j \neq x_i + x_j$. Then, without loss of generality, there is $k \in K$ such that $z_i^k + z_j^k > x_i^k + x_j^k$. By Lemma 1,

$$z^k_{\{i,j\}} = \phi^k(R_i, R_j, z_i + z_j; u) \geq \phi^k(R_i, R_j, x_i + x_j; u) = x^k_{\{i,j\}}$$

and, without loss of generality, $z_i^k > x_i^k$. Thus, since $\sum_{h\in N} x_h^k = m^k = \sum_{h\in N} z_h^k$, there is $l \in N \setminus \{i, j\}$ such that $z_l^k < x_l^k$. By the *consistency* of $\phi(\cdot\,; u)$, $x_{\{i,l\}} = \phi(R_i, R_l, x_i + x_l; u)$ and, by assumption, $z_{\{i,l\}} = \phi(R_i, R_l, z_i + z_l; u)$. By Lemma 1, if $z_i^k + z_l^k \geq x_i^k + x_l^k$, then $z_l^k \geq x_l^k$, which is not the case. Thus, $z_i^k + z_l^k < x_i^k + x_l^k$. Thus, by Lemma 1, $z^k_{\{i,l\}} \leq x^k_{\{i,l\}}$, contradicting $z_i^k > x_i^k$. Thus, $\phi(\cdot\,; u)$ is *conversely consistent*. □

A.3. Proof of theorems

The first step in the proofs is showing that some of the properties of mechanisms discussed thus far imply the following condition:

Same-sidedness: For each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each $k \in K$,
$\sum_{i \in N} p^k(R_i) \leq m^k$ implies $\varphi^k(R, m) \geq p^k(R)$ and
$\sum_{i \in N} p^k(R_i) \geq m^k$ implies $\varphi^k(R, m) \leq p^k(R)$.

Lemma 3. *A strategy-proof, unanimous, and non-bossy mechanism is same-sided.*

The proof of Lemma 3 is almost identical to that of Lemma 1 in Morimoto et al. (2013). (For completeness, the proof is included in a supplementary note.) We now prove that some properties of mechanisms discussed thus far imply a useful informational simplicity condition:

Peaks-only: For each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each $R' \in \mathcal{R}^N$, $p(R') = p(R)$ implies $\varphi(R, m) = \varphi(R', m)$.

Lemma 4. *A strategy-proof, same-sided, non-bossy, and resource-monotonic mechanism is peaks-only.*

Proof. Let φ denote a mechanism satisfying the properties in Lemma 4. Let $N \in \mathcal{N}$. For each $(R, m) \in \mathcal{E}^N$, let

$$\mathcal{K}(R, m) \equiv \{k \in K : m^k \neq \textstyle\sum_{i \in N} p^k(R_i), m^k \neq 0, m^k \neq \sum_{i \in N} c_i^k\}.$$

The proof of Lemma 4 is by induction on the cardinality of $\mathcal{K}(R, m)$. Claim 1 below establishes the induction basis and Claim 2 below establishes the inductive step.

Claim 1. *Let $(R, m) \in \mathcal{E}^N$ be such that $\mathcal{K}(R, m)$ is empty and thus has a cardinality of zero. Then, for each $\tilde{R} \in \mathcal{R}^N$ such that $p(R) = p(\tilde{R})$, $\varphi(R, m) = \varphi(\tilde{R}, m)$.*

Let $(R, m) \in \mathcal{E}^N$ be such that $\mathcal{K}(R, m)$ is empty and let $\tilde{R} \in \mathcal{R}^N$ be such that $p(R) = p(\tilde{R})$. Let $x \equiv \varphi(R, m)$ and $y \equiv \varphi(\tilde{R}, m)$. We need to prove that $x = y$.

Because $p(R) = p(\tilde{R})$, $\mathcal{K}(\tilde{R}, m)$ is also empty. Thus, for each $k \in K$, at least one of the below cases is true:

(i) $m^k = \sum_{i \in N} p^k(R_i) = \sum_{i \in N} p^k(\tilde{R}_i)$. By *same-sidedness*, $x^k = p^k(R)$ and $y^k = p^k(\tilde{R})$. However, $p^k(R) = p^k(\tilde{R})$. Thus, $x^k = y^k$.
(ii) $m^k = 0$. By feasibility, for each $i \in N$, $x_i^k = 0 = y_i^k$.
(iii) $m^k = \sum_{i \in N} c_i^k$. By feasibility, for each $i \in N$, $x_i^k = c_i^k = y_i^k$.

Thus, $x = y$.

Claim 2. *Let $n \in \{0, 1, \ldots, |K| - 1\}$.*
Suppose that, for each $(R, m) \in \mathcal{E}^N$, $|\mathcal{K}(R, m)| \leq n$ implies that, for each $\tilde{R} \in \mathcal{R}^N$ with $p(\tilde{R}) = p(R)$, $\varphi(R, m) = \varphi(\tilde{R}, m)$.
Then, for each $(R, m) \in \mathcal{E}^N$, $|\mathcal{K}(R, m)| = n + 1$ implies that, for each $\tilde{R} \in \mathcal{R}^N$ with $p(\tilde{R}) = p(R)$, $\varphi(R, m) = \varphi(\tilde{R}, m)$.

Let $(R, m) \in \mathcal{E}^N$ be such that $|\mathcal{K}(R, m)| = n + 1$, let $k \in \mathcal{K}(R, m)$, and let $p \equiv p(R)$. Let $w \in M(N)$ be such that $w^k = \sum_{i \in N} p_i^k$ and, for each $l \in K \setminus \{k\}$, $w^l = m^l$. Let $x \equiv \varphi(R, w)$.

Then, for each $R' \in \mathcal{R}^N$ such that $p(R') = p$, $\mathcal{K}(R, w)$ and $\mathcal{K}(R', w)$ coincide and have cardinality n. Thus, by the inductive assumption,

$$\text{for each } R' \in \mathcal{R}^N \text{ such that } p(R') = p, \quad \varphi(R', w) = x. \tag{2}$$

Thus, since w is between m and p, by *resource-monotonicity*,

$$\text{for each } R' \in \mathcal{R}^N \text{ such that } p(R') = p \text{ and each } i \in N,\ x_i\ R'_i\ \varphi_i(R', m). \tag{3}$$

Next, we prove that,

$$\text{for each } R' \in \mathcal{R}^N \text{ such that } p(R') = p \text{ and each } l \in K \setminus \{k\}, \quad \varphi^l(R', m) = x^l. \tag{4}$$

Let $R' \in \mathcal{R}^N$ be such that $p(R') = p$ and $y \equiv \varphi(R', m)$. To establish (4), we first prove that,

$$\begin{aligned} \text{for each } l \in K \setminus \{k\}, \quad & \text{(i) if } m^l \le \textstyle\sum_{h \in N} p_h^l, \text{ for each } i \in N,\ y_i^l \le x_i^l \le p_i^l; \\ & \text{(ii) if } m^l > \textstyle\sum_{h \in N} p_h^l, \text{ for each } i \in N,\ y_i^l \ge x_i^l \ge p_i^l. \end{aligned} \tag{5}$$

By way of contradiction suppose that there is $l \in K \setminus \{k\}$ such that (5) fails. Then, without loss of generality, suppose that (i) in (5) fails. By *same-sidedness*, for each $i \in N$, $y_i^l, x_i^l \le p_i^l$. Thus, there is $i \in N$ such that

$$x_i^l < y_i^l \le p_i^l. \tag{6}$$

Let $\hat{R}_i \in \mathcal{R}_i$ denote a preference relation with the following utility representation:

$$\text{for each } \chi_i \in X_i, \quad U_i(\chi_i) \equiv -\textstyle\sum_{h \in K} a_i^h (\chi_i^h - p_i^h)^2$$

where each a_i^h is a strictly positive number and note that $p(\hat{R}_i) = p_i$; moreover, by (6), a_i^l can be chosen so that $y_i\ \hat{P}_i\ x_i$. On the other hand, by (3), since $p(\hat{R}_i, R'_{-i}) = p$, $x_i\ \hat{R}_i\ \varphi_i(\hat{R}_i, R'_{-i}, m)$. Thus, $y_i\ \hat{P}_i\ x_i\ \hat{R}_i\ \varphi_i(\hat{R}_i, R'_{-i}, m)$. Thus, $y_i\ \hat{P}_i\ \varphi_i(\hat{R}_i, R'_{-i}, m)$ and, since $y_i = \varphi_i(R', m)$,

$$\varphi_i(R', m)\ \hat{P}_i\ \varphi_i(\hat{R}_i, R'_{-i}, m)$$

violating *strategy-proofness*. This contradiction establishes (5).

In case (i) in (5), since $\sum_{i \in N} y_i^l = m^l = w^l = \sum_{i \in N} x_i^l$ and, for each $i \in N$, $y_i^l \le x_i^l$, in fact, $x_i^l = y_i^l$. In case (ii) in (5), since $m^l = w^l$ and, for each $i \in N$, $y_i^l \ge x_i^l$, in fact, $x_i^l = y_i^l$. This establishes (4).

To conclude the proof, let $\tilde{R} \in \mathcal{R}^N$ be such that $p(\tilde{R}) = p$. First, suppose that $m^k \le \sum_{i \in N} p_i^k$. Label N so that $N = \{1, \dots, h\}$. Since $p(R) = p(\tilde{R}_1, R_{N \setminus \{1\}})$, by (4),

$$\text{for each } l \in K \setminus \{k\}, \quad \varphi^l(R, m) = x^l = \varphi^l(\tilde{R}_1, R_{N \setminus \{1\}}, m). \tag{7}$$

By *strategy-proofness*,

$$\varphi_1(R, m)\ R_1\ \varphi_1(\tilde{R}_1, R_{N \setminus \{1\}}, m) \text{ and } \varphi_1(\tilde{R}_1, R_{N \setminus \{1\}}, m)\ \tilde{R}_1\ \varphi_1(R, m).$$

By *same-sidedness* and (7), these expressions imply $\varphi_1^k(R, m) \ge \varphi_1^k(\tilde{R}_1, R_{N \setminus \{1\}}, m)$ and $\varphi_1^k(R, m) \le \varphi_1^k(\tilde{R}_1, R_{N \setminus \{1\}}, m)$, respectively. Thus, $\varphi_1^k(R, m) = \varphi_1^k(\tilde{R}_1, R_{N \setminus \{1\}}, m)$. Thus, by (7), $\varphi_1(R, m) = \varphi_1(\tilde{R}_1, R_{N \setminus \{1\}}, m)$. By *non-bossiness*, $\varphi(R, m) = \varphi(\tilde{R}_1, R_{N \setminus \{1\}}, m)$. Repeating these arguments $h - 1$ more times we find that

$$\varphi(R, m) = \varphi(\tilde{R}_1, R_{N\setminus\{1\}}, m) = \varphi(\tilde{R}_1, \tilde{R}_2, R_{N\setminus\{1,2\}}, m) = \cdots = \varphi(\tilde{R}, m).$$

An analogous proof establishes the same conclusion when $m^k \geq \sum_{i\in N} p_i^k$. □

Lemma 5. *A strategy-proof, same-sided, resource-monotonic, and peaks-only mechanism satisfies physical resource-monotonicity.*

Proof. Let φ denote a mechanism satisfying the properties in Lemma 5 and let $N \in \mathcal{N}$. We first prove the following statement:

$$\text{for each } k \in K, \text{ each } (R, m) \in \mathcal{E}^N, \text{ and each } \tilde{m} \in M(N) \text{ such that,}$$
$$m^k \leq \tilde{m}^k \text{ and, for each } l \in K \setminus \{k\},\ \tilde{m}^l = m^l,\ \varphi(R, m) \leq \varphi(R, \tilde{m}). \tag{8}$$

Let $k \in K$, $(R, m) \in \mathcal{E}^N$, and $\tilde{m} \in M(N)$ be as specified in (8). Let $p \equiv p(R)$, $x \equiv \varphi(R, m)$, and $y \equiv \varphi(R, \tilde{m})$. There are three possible cases:

Case 1. $m^k \leq \tilde{m}^k \leq \sum_{i\in N} p_i^k$**.** By *peaks-only*,

$$\text{for each } \tilde{R} \in \mathcal{R}^N \text{ such that } p(\tilde{R}) = p,\ \varphi(\tilde{R}, m) = x \text{ and } \varphi(\tilde{R}, \tilde{m}) = y.$$

Thus, since $\tilde{m}$ is between m and p, by *resource-monotonicity*,

$$\text{for each } \tilde{R} \in \mathcal{R}^N \text{ such that } p(\tilde{R}) = p, \text{ for each } i \in N,\ y_i\ \tilde{R}_i\ x_i. \tag{9}$$

We now prove that

$$\text{for each } l \in K \setminus \{k\}, \quad y^l = x^l. \tag{10}$$

Otherwise there are $l \in K \setminus \{k\}$ and $i \in N$ such that $y_i^l < x_i^l$ because $m^l = \tilde{m}^l$. By *same-sidedness*, $y_i^l < x_i^l \leq p_i^l$. Let $\hat{R}_i \in \mathcal{R}_i$ denote a preference relation with the following utility representation:

$$\text{for each } \chi_i \in X_i, \quad U_i(\chi_i) \equiv -\textstyle\sum_{h\in K} a_i^h (\chi_i^h - p_i^h)^2$$

where each a_i^h is a strictly positive number and note that $p(\hat{R}_i) = p_i$; moreover, since $y_i^l < x_i^l \leq p_i^l$, a_i^l can be chosen so that $x_i\ \hat{P}_i\ y_i$. Note that $p(\hat{R}_i, R_{-i}) = p$ and, thus, by (9), $y_i\ \hat{R}_i\ x_i$. This contradiction establishes (10).

It remains to prove that $x^k \leq y^k$. Otherwise, because $m^k \leq \tilde{m}^k$, there is $i \in N$ such that $y_i^k < x_i^k$. By *same-sidedness*, $y_i^k < x_i^k \leq p_i^k$. Thus, by (10) and multidimensional single-peakedness, $x_i\ P_i\ y_i$. This contradicts *resource-monotonicity* and thus $x^k \leq y^k$. Thus, by (10), $x \leq y$.

Case 2. $\sum_{i\in N} p_i^k \leq m^k \leq \tilde{m}^k$**.** Note that m is between $\tilde{m}$ and p so *resource-monotonicity* applies. The proof that $x \leq y$ is thus analogous to that in Case 1.

Case 3. $m^k \leq \sum_{i\in N} p_i^k \leq \tilde{m}^k$**.** Let $\hat{m} \in M(N)$ be such that, for each $l \in K \setminus \{k\}$, $\tilde{m}^l = m^l$, and $\hat{m}^k = \sum_{i\in N} p_i^k$. By Case 1, $x \leq \varphi(R, \hat{m})$. By Case 2, $\varphi(R, \hat{m}) \leq y$. Altogether, $x \leq y$. This concludes the proof of (8).

To establish that φ is *physically resource-monotonic* we repeatedly apply (8). Let $(R, m) \in \mathcal{E}^N$ and $\tilde{m} \in M(N)$ be such that $\tilde{m} \geq m$.

Label K so that $K = \{1, \ldots, |K|\}$. For each $k \in K$, let $m_k \in M(N)$ be such that,

$$m_k^l = \begin{cases} \tilde{m}^l & \text{if } l \leq k, \\ m^l & \text{if } l > k. \end{cases}$$

Then, since $\tilde{m} \geq m$, $\tilde{m} = m_{|K|} \geq m_{|K|-1} \geq \cdots \geq m_1 \geq m$. Then, by (8),

$$\varphi(R, \tilde{m}) = \varphi(R, m_{|K|}) \geq \varphi(R, m_{|K|-1}) \geq \cdots \geq \varphi(R, m_1) \geq \varphi(R, m). \qquad \square$$

Lemma 6. *Let φ denote a strategy-proof, same-sided, and non-bossy mechanism satisfying physical resource-monotonicity. For each $k \in K$ and each pair $(R, m), (\tilde{R}, \tilde{m})$ in $\mathcal{E}^N$ such that $p^k(R) = p^k(\tilde{R})$ and $m^k = \tilde{m}^k$, $\varphi^k(R, m) = \varphi^k(\tilde{R}, \tilde{m})$.*

Proof. Let φ denote a mechanism satisfying the properties in Lemma 6. Let $k \in K$ and let $(R, m), (\tilde{R}, \tilde{m}) \in \mathcal{E}^N$ be such that $p^k(R) = p^k(\tilde{R})$ and $m^k = \tilde{m}^k$. Let $x \equiv \varphi(R, m)$ and $y \equiv \varphi(\tilde{R}, \tilde{m})$. We will prove that $x^k = y^k$.

Let $w \in M(N)$ be such that $w^k = m^k$ and, for each $l \in K \setminus \{k\}$, $w^l = 0$. Let $a \equiv \varphi(R, w)$ and $b \equiv \varphi(\tilde{R}, w)$. Since $w \leq m$ and $w \leq \tilde{m}$, by *physical resource-monotonicity*, $a \leq x$ and $b \leq y$. Thus, since $\sum_{i \in N} a_i^k = m^k = \sum_{i \in N} b_i^k$,

$$a^k = x^k \quad \text{and} \quad b^k = y^k. \tag{11}$$

Thus, it suffices to show that $a^k = b^k$. Note that, by feasibility,

$$\text{for each } \hat{R} \in \mathcal{R}^N, \text{ each } l \in K \setminus \{k\}, \text{ and each } i \in N,\ \varphi_i^l(\hat{R}, w) = 0. \tag{12}$$

Suppose that $m^k \leq \sum_{i \in N} p^k(R_i)$. Label N so that $N = \{1, \ldots, n\}$. Thus, by *strategy-proofness*, $\varphi_1(R, w)\ R_1\ \varphi_1(\tilde{R}_1, R_{N \setminus \{1\}}, w)$ and $\varphi_1(\tilde{R}_1, R_{N \setminus \{1\}}, w)\ \tilde{R}_1\ \varphi_1(R, w)$. By *same-sidedness* and (12), the first expression implies $\varphi_1^k(R, w) \geq \varphi_1^k(\tilde{R}_1, R_{N \setminus \{1\}}, w)$, whereas the second expression implies $\varphi_1^k(R, w) \leq \varphi_1^k(\tilde{R}_1, R_{N \setminus \{1\}}, w)$. Thus, $\varphi_1^k(R, w) = \varphi_1^k(\tilde{R}_1, R_{N \setminus \{1\}}, w)$. Thus, by (12), $\varphi_1(R, w) = \varphi_1(\tilde{R}_1, R_{N \setminus \{1\}}, w)$. By *non-bossiness*, $\varphi(R, w) = \varphi(\tilde{R}_1, R_{N \setminus \{1\}}, w)$. Repeating these arguments $n - 1$ more times we find that

$$a = \varphi(R, w) = \varphi(\tilde{R}_1, R_{N \setminus \{1\}}, w) = \varphi(\tilde{R}_1, \tilde{R}_2, R_{N \setminus \{1,2\}}, w) = \cdots = \varphi(\tilde{R}, w) = b.$$

An analogous proof establishes the same conclusion when $m^k \geq \sum_{i \in N} p^k(R_i)$. $\square$

Next, we use Lemma 6 to “decompose” our multidimensional allocation problem into $|K|$ uni-dimensional allocation problems. The result is reminiscent of the decomposition of strategy-proof social choice functions into “marginal” strategy-proof social choice functions in problems where preferences have some degree of separability over a set of alternatives with a product structure (Barberà et al., 1993; Le Breton and Sen, 1999).

Lemma 7. *Let φ denote a mechanism satisfying strategy-proofness, same-sidedness, non-bossiness, and physical resource-monotonicity. Then, φ is separable.*

Proof. This follows immediately from Lemma 6. $\square$

Lemma 8. *Let φ denote a physically resource-monotonic and separable mechanism. Then, for each $N \in \mathcal{N}$ and each $R \in \mathcal{R}^N$, $\varphi(R, \cdot)$ is continuous.*

Proof. Let φ denote a mechanism satisfying the properties in Lemma 8. By *separability*, φ has a decomposition $\{\psi^k : k \in K\}$ in Ψ. Let $N \in \mathcal{N}$, and $(R, m) \in \mathcal{E}^N$. Let $k \in K$ and note that it suffices to prove that $\psi^k(p^k(R), \cdot)$ is continuous.

Let $\{\mu_n\}_{n\in\mathbb{N}}$ denote a sequence in $M^k(N)$ converging to m^k and, for each $n \in \mathbb{N}$, let $x(n) \equiv \psi^k(p^k(R), \mu_n)$. We now prove that $\{x(n)\}_{n\in\mathbb{N}}$ has a limit x and that $x = y \equiv \psi^k(p^k(R), m^k)$ which will establish the desired conclusion. By the Cauchy convergence criterion, if $\{x(n)\}_{n\in\mathbb{N}}$ does not have a limit, there is an $\varepsilon > 0$ such that, for each natural number ν, there are natural numbers $h, l \geq \nu$ for which $\|x(l) - x(h)\| \geq \varepsilon$. Note that this requires that $\mu_l \neq \mu_h$ for otherwise, by definition, $x(l) = x(h)$. Without loss of generality, suppose that $\mu_h > \mu_l$. Then, by Lemma 1, $x(h) \geq x(l)$. However, since $\{\mu_n\}_{n\in\mathbb{N}}$ converges, ν can be chosen sufficiently large so that $\mu_h - \mu_l < \varepsilon$. Clearly, this is incompatible with $x(h) \geq x(l)$ and $\|x(l) - x(h)\| \geq \varepsilon$, a contradiction. Thus, $\{x(n)\}_{n\in\mathbb{N}}$ has a limit which we denote by x. Since $\{\mu_n\}_{n\in\mathbb{N}}$ converges to m^k and $x(n) \equiv \psi^k(p^k(R), \mu_n)$, $\{\sum_{i\in N} x_i(n)\}_{n\in\mathbb{N}}$ converges to m^k. Thus, since $\{x(n)\}_{n\in\mathbb{N}}$ converges to x, $\sum_{i\in N} x_i = m^k$. By way of contradiction, suppose that $x \neq y$. Then, since $\sum_{i\in N} x_i = m^k = \sum_{i\in N} y_i$, there is a pair $i, j \in N$ such that $x_i > y_i$ and $x_j < y_j$. Then, since $\{x(n)\}_{n\in\mathbb{N}}$ converges to x, there is a sufficiently large $n \in \mathbb{N}$ such that $x_i(n) > y_i$ and $x_j(n) < y_j$. If $\mu_n \geq m^k$, by Lemma 1, $x(n) \equiv \psi^k(p^k(R), \mu_n) \geq \psi^k(p^k(R), m^k) \equiv y$, a contradiction. If $\mu_n < m^k$, by Lemma 1, $x(n) \equiv \psi^k(p^k(R), \mu_n) \leq \psi^k(p^k(R), m^k) \equiv y$, a contradiction again. Thus, $x = y$. □

Lemma 9. *Let φ denote a mechanism satisfying physically resource-monotonicity and separability. If $\{\psi^k : k \in K\}$ in Ψ denotes the decomposition of φ, then, for each $k \in K$, ψ^k satisfies the following properties:*

(*i*) *For each $i \in A$, there is a function $u_i^{xd,k} : X_i^k \to \mathbb{R}$ that is strictly concave, continuous, and such that, for each $m^k \in M^k(A)$,*

$$\psi^k((c_i^k)_{i\in A}, m^k) = \arg\max \left\{ \textstyle\sum_{i\in A} u_i^{xd,k}(z_i) : \sum_{i\in A} z_i = m^k, z \in \times_{i\in A} X_i^k \right\}.$$

(*ii*) *For each $i \in A$, there is a function $u_i^{xs,k} : X_i^k \to \mathbb{R}$ that is strictly concave, continuous, and such that, for each $m^k \in M^k(A)$,*

$$\psi^k((0)_{i\in A}, m^k) = \arg\max \left\{ \textstyle\sum_{i\in A} u_i^{xs,k}(z_i) : \sum_{i\in A} z_i = m^k, z \in \times_{i\in A} X_i^k \right\}.$$

Proof. Let φ denote a mechanism satisfying the properties in Lemma 9. Let $\{\psi^k : k \in K\}$ in Ψ denote the decomposition of φ. We prove statement (i) in the lemma. The proof of statement (ii) is symmetric. Let $k \in K$, $c \equiv (c_i^k)_{i\in A}$, $C \equiv \sum_{i\in A} c_i^k$, and $X \equiv \times_{i\in A} X_i^k$. Without loss of generality, we can assume that, for each $i \in A$, the interior of X_i^k relative to $\mathbb{R}$ is non-empty.[11]

Step 1. *Constructing a continuous monotone path $g : [0, C] \to X$.*

For each $m \in [0, C]$, let $g(m) \equiv \psi^k(c, m)$. By Lemma 1, for each pair $m, m' \in [0, C]$, $m' \geq m$ implies

[11] Otherwise we could replace A by $A^k \equiv \{i \in A : c_i^k \neq 0\}$ in all the arguments in the proof and attribute, to each $i \in A \setminus A^k$ any finite function $u_i^{xd,k}$ with domain $X_i^k = \{0\}$.

$$g(m') = \psi^k(c, m') \geq \psi^k(c, m) = g(m).$$

Thus, for each $i \in A$, g_i is non-decreasing in m. By feasibility, for each $i \in A$, $g_i(0) = 0$ and $g_i(C) = c_i^k$. Moreover, by Lemma 8, g is continuous on $[0, C]$.

Step 2. *Constructing $(u_i^{xd,k})_{i\in A}$ from the monotone path g.*

For each $i \in A$, let $h_i : X_i^k \to [0, C]$ denote a strictly increasing function such that,

$$\begin{aligned}
&\text{for each } m \in [0, C],\\
&\text{if } x_i \in (0, c_i^k), \quad x_i = g_i(m) \quad \text{if and only if} \quad \lim_{z\uparrow x_i} h_i(z) \leq m \leq \lim_{z\downarrow x_i} h_i(z),\\
&\qquad\qquad\qquad\quad 0 = g_i(m) \quad \text{if and only if} \quad 0 \leq m \leq \lim_{z\downarrow 0} h_i(z), \text{ and}\\
&\qquad\qquad\qquad\quad c_i^k = g_i(m) \quad \text{if and only if} \quad \lim_{z\uparrow c_i^k} h_i(z) \leq m \leq C.
\end{aligned} \tag{13}$$

By Theorem 6.9 in Rudin (1976), h_i is Riemann-integrable on $X_i^k = [0, c_i^k]$ and on any sub-interval. For each $x_i \in X_i^k = [0, c_i^k]$, let $f_i(x_i)$ denote the Riemann integral $\int_0^{x_i} h_i(t)\mathrm{d}t$. Note that $f_i : X_i^k \to \mathbb{R}$ thus defined is continuous on X_i^k since, by the properties of the Riemann integral (see Theorem 6.12 in Rudin, 1976) and the fact that $h_i \leq C$, for each pair $\alpha, \beta \in X_i^k$, $|f_i(\beta) - f_i(\alpha)| = |\int_\alpha^\beta h_i(t)\mathrm{d}t| \leq |\beta - \alpha|C$. Additionally, because h_i is strictly increasing, f_i is strictly convex. For each $i \in A$, let $u_i^{xd,k} \equiv -f_i$. Hence, each $u_i^{xd,k} : X_i^k \to \mathbb{R}$ is strictly concave and continuous.

Step 3. *Verifying that $(u_i^{xd,k})_{i\in A}$ is as claimed in (i) of Lemma 9.*

For each $i \in A$, let $f_i \equiv -u_i^{xd,k}$. It suffices to establish that, for each $m \in [0, C]$,

$$g(m) = \arg\min \left\{\textstyle\sum_A f_i(z_i) : z \in X, \sum_A z_i = m\right\}. \tag{14}$$

Case 1. $m = 0$ or $m = C$. If $m = 0$, $\{z \in X : \sum_A z_i = m\}$ is the singleton $\{(0)_{i\in A}\}$. If $m = C$, $\{z \in X : \sum_A z_i = m\}$ is the singleton $\{c\}$. Thus, (14) follows because $g(0) = (0)_{i\in A}$ and $g(C) = c$.

Case 2. $0 < m < C$. Let $a \equiv \arg\min \left\{\sum_A f_i(z_i) : z \in X, \sum_A z_i = m\right\}$. For each $i \in A$, let $F_i : \mathbb{R} \to [-\infty, +\infty]$ be the function such that, for each $x_i \in X_i^k$, $F_i(x_i) = f_i(x_i)$ and, for each $x_i \notin X_i^k$, $F_i(x_i) = \infty$. Note that, under the standard convention that the convex combination of a finite number and ∞ is itself ∞, F_i is convex, closed, and proper. Moreover,

$$a = \arg\min \left\{\textstyle\sum_A F_i(z_i) : \sum_A z_i = m\right\}.$$

Clearly, there is x in the relative interior of X such that $\sum_A x_i = m$ and $\sum_A F_i(x_i) = \sum_A f_i(x_i) \neq -\infty$. Thus, by Corollary 28.2.2 in Rockafellar (1970), there is a Kuhn–Tucker coefficient $\lambda^* \in \mathbb{R}$ for the optimization problem $\min\left\{\sum_A F_i(z_i) : \sum_A z_i = m\right\}$. For each $(x, \lambda) \in \mathbb{R}^A \times \mathbb{R}$, let $L(x, \lambda) \equiv \sum_A F_i(x_i) + \lambda[m - \sum_A x_i]$. By Theorem 28.3 in Rockafellar (1970),

$$\begin{aligned}\min_{x\in\mathbb{R}^A} L(x,\lambda^*) &= \lambda^* m + \sum_{i\in A} \min\left\{F_i(x_i) - \lambda^* x_i : x_i \in \mathbb{R}\right\}\\ &= \lambda^* m + \sum_{i\in A}\{F_i(a_i) - \lambda^* a_i\}.\end{aligned}$$

Thus, for each $i \in A$ and each $x_i \in \mathbb{R}$,

$$F_i(x_i) \geq F_i(a_i) + \lambda^*(x_i - a_i).$$

Thus, λ^* is in the sub-differential of F_i at a_i. That is, for each $i \in A$, $\lambda^* \in \partial F_i(a_i)$. By the definition of f_i in Step 2, the definition of F_i, and Theorem 24.2 in Rockafellar (1970),

$$\begin{aligned}\text{for each } i \in A,\quad &\text{(i)}\quad \text{if } a_i \in (0, c_i^k),\ \partial F_i(a_i) = [\lim_{z\uparrow a_i} h_i(z), \lim_{z\downarrow a_i} h_i(z)],\\ &\text{(ii)}\quad \text{if } a_i = 0\ \partial F_i(a_i) = \left(-\infty, \lim_{z\downarrow a_i} h_i(z)\right],\\ &\text{(iii)}\quad \text{if } a_i = c_i^k,\ \partial F_i(a_i) = \left[\lim_{z\uparrow a_i} h_i(z), \infty\right).\end{aligned}$$

Moreover, since $0 < m < C$, there are $i, j \in A$ such that $0 < a_j$ and $a_i < c_i^k$. Thus, since h_j is strictly increasing, $0 \leq h_j(0) < \lim_{z\uparrow a_j} h_j(z)$. Thus, by (i) and (iii), $0 < \lambda^*$. Similarly, since h_i is strictly increasing, $\lim_{z\downarrow a_i} h_i(z) < h_i(c_i^k) \leq C$. Thus, by (i) and (ii), $\lambda^* < C$. Thus, $0 < \lambda^* < C$. Thus,

$$\begin{aligned}\text{for each } i \in A,\quad &\text{by (i), if } a_i \in (0, c_i^k),\ \lambda^* \in [\lim_{z\uparrow a_i} h_i(z), \lim_{z\downarrow a_i} h_i(z)],\\ &\text{by (ii), if } a_i = 0,\ \lambda^* \in \left(0, \lim_{z\downarrow a_i} h_i(z)\right],\\ &\text{by (iii), if } a_i = c_i^k,\ \lambda^* \in \left[\lim_{z\uparrow a_i} h_i(z), C\right).\end{aligned}$$

Thus, by (13), for each $i \in A$, $g_i(\lambda^*) = a_i$. Thus, $m = \sum_A a_i = \sum_A g_i(\lambda^*) = \lambda^*$. Thus, $m = \lambda^*$ and $g(m) = a$, confirming (14). □

Lemma 10. *A consistent mechanism is non-bossy.*

Proof. Let φ denote a *consistent* mechanism. Let $N \in \mathcal{N}$, $(R, m) \in \mathcal{E}^N$, and $i \in N$. Let $R' \in \mathcal{R}^N$ be such that, for each $j \in N \setminus \{i\}$, $R'_j = R_j$. Let $x \equiv \varphi(R, m)$ and $x' \equiv \varphi(R', m)$. Suppose, as in the hypothesis of *non-bossiness*, that $x'_i = x_i$. Then, $\sum_{N\setminus\{i\}} x'_j = \sum_{N\setminus\{i\}} x_j$. Thus, $(R'_{N\setminus\{i\}}, \sum_{N\setminus\{i\}} x'_j) = (R_{N\setminus\{i\}}, \sum_{N\setminus\{i\}} x_j)$. Thus, by *consistency*, for each $j \in N \setminus \{i\}$, $x'_j = \varphi_j(R_{N\setminus\{i\}}, \sum_{N\setminus\{i\}} x_j) = x_j$. Thus, $x' = x$. □

Lemma 11. *Let φ denote a mechanism satisfying strategy-proofness, consistency, physical resource-monotonicity, and same-sidedness.*

Then, there is u in $\mathcal{U}$ such that, for each $N \in \mathcal{N}$ ***consisting of two agents*** *and each $(R, m) \in \mathcal{E}^N$, $\varphi(R, m) = \phi(R, m; u)$.*

Proof. Let φ denote a mechanism satisfying the properties in Lemma 11. By Lemma 10, φ is *non-bossy*. Thus, by Lemma 7, φ is *separable*. Thus, there is a decomposition $\{\psi^k : k \in K\}$ in Ψ of φ. Thus, by Lemma 9, there is

$$u = \{(u_i^{xd,k}, u_i^{xs,k}) : i \in A, k \in K\} \text{ in } \mathcal{U}$$

such that, for each $k \in K$, each $i \in A$, and each $\mu \in M^k(A)$,

(i) $\psi^k((c_i^k)_{i\in A}, \mu) = \arg\max\{\sum_{i\in A} u_i^{xd,k}(z_i) : \sum_{i\in A} z_i = \mu, z \in \times_{i\in A} X_i^k\}$, and
(ii) $\psi^k((0)_{i\in A}, \mu) = \arg\max\{\sum_{i\in A} u_i^{xs,k}(z_i) : \sum_{i\in A} z_i = \mu, z \in \times_{i\in A} X_i^k\}$.

Let $\{i, j\} \in \mathcal{N}$ and $(R_i, R_j, m) \in \mathcal{E}^{\{i,j\}}$. We prove $\varphi(R_i, R_j, m) = \phi(R_i, R_j, m; u)$. Let $k \in K$, $r_i \equiv p^k(R_i)$, and $r_j \equiv p^k(R_j)$.

First, consider the case where $r_i + r_j \geq m^k$. Then, it suffices to prove that

$$\begin{aligned}\varphi^k(R_i, R_j, m) &= \arg\max\{u_i^{xd,k}(z_i) + u_j^{xd,k}(z_j) : z \in S^k((r_i, r_j), m^k)\} \\ &= \phi^k(R_i, R_j, m; u) \qquad (15)\end{aligned}$$

where the notation $S^k(\cdot, m^k)$ is defined in Section A.1. The second equality in (15) follows immediately from the definition of $\phi(\cdot; u)$. It remains to establish the first equality in (15).

Let $\bar{R} \in \mathcal{R}^A$ be such that $p^k(\bar{R}) = (c_i^k)_{i\in A}$ and, for each $l \in K \setminus \{k\}$, $p^l(\bar{R}_i) = p^l(R_i)$ and $p^l(\bar{R}_j) = p^l(R_j)$. Since, as noted above φ is *separable*, Lemma 8 implies there is $\bar{m} \in M(A)$ such that, $\varphi_i(\bar{R}, \bar{m}) + \varphi_j(\bar{R}, \bar{m}) = m$. Then,

$$\varphi^k(\bar{R}, \bar{m}) = \psi^k((c_i^k)_{i\in A}, \bar{m}^k) = \phi^k(\bar{R}, \bar{m}; u)$$

where the first equality follows from the definition of ψ^k and the second equality follows from (i) and the definition of $\phi(\cdot; u)$. Thus, by the assumed *consistency* of φ and that of $\phi(\cdot; u)$, as established in Lemma 2,

$$\begin{aligned}\psi^k(c_i^k, c_j^k, m^k) &= \phi^k(\bar{R}_i, \bar{R}_j, m; u) \\ &= \arg\max\{u_i^{xd,k}(z_i) + u_j^{xd,k}(z_j) : z_i + z_j = m^k, z_i \in X_i^k, z_j \in X_j^k\}.\end{aligned}$$

Let $x \equiv \varphi(\bar{R}_i, \bar{R}_j, m)$, $y \equiv \varphi(R_i, R_j, m)$, $p_i \equiv p^k(\bar{R}_i)$, and $p_j \equiv p^k(\bar{R}_j)$.

Before proceeding, note that, by the strict concavity of $u_i^{xd,k} + u_j^{xd,k}$, for each $\alpha \in \mathbb{R}$, $U(\alpha) \equiv \{(a_i, a_j) \in X_i^k \times X_j^k : u_i^{xd,k}(a_i) + u_j^{xd,k}(a_j) \geq \alpha\}$ is strictly convex relative to $X_i^k \times X_j^k$. Note that, since $r_i \leq p_i$ and $r_j \leq p_j$,

$$S^k((r_i, r_j), m^k) \subseteq S^k((r_i, p_j), m^k) \subseteq S^k((p_i, p_j), m^k).$$

Thus, $x^k \in S^k((r_i, r_j), m^k)$ would require $\phi^k(R_i, R_j, m; u) = x^k$ since, for each z in $S^k((r_i, r_j), m^k)$, $u_i^{xd,k}(x_i^k) + u_j^{xd,k}(x_j^k) \geq u_i^{xd,k}(z_i) + u_j^{xd,k}(z_j)$; moreover, by the strict convexity of $U(\cdot)$, this weak inequality would be an equality only if $z = x^k$. On the other hand, if x^k is not in $S^k((r_i, r_j), m^k)$, then, by the strict convexity of $U(\cdot)$, $\phi^k(R_i, R_j, m; u)$ is the closest point to x^k in $S^k((r_i, r_j), m^k)$. Thus, to prove the first equality in (15) we distinguish the following two cases:

Case 1. $x^k \in S^k((r_i, r_j), m^k)$. Let $w \equiv \varphi(R_i, \bar{R}_j, m)$. As noted in the beginning of the proof, φ is *separable*. Thus, since, for each $l \in K \setminus \{k\}$, $p^l(\bar{R}_i) = p^l(R_i)$ and $p^l(\bar{R}_j) = p^l(R_j)$, $w^l = x^l$ and $y^l = x^l$. Thus, suppose that $w^k \neq x^k$. If $x_i^k < w_i^k$, by *same-sidedness*, $x_i^k < w_i^k \leq p^k(R_i) \leq p^k(\bar{R}_i)$. Then, by multidimensional single-peakedness, $w_i \; \bar{P}_i \; x_i$, contradicting the *strategy-proofness* of φ at $(\bar{R}_i, \bar{R}_j, m)$. If $x_i^k > w_i^k$, $x^k \in S^k((r_i, r_j), m^k)$ implies $w_i^k < x_i^k \leq p^k(R_i)$. Then, by multidimensional single-peakedness, $x_i \; P_i \; w_i$, contradicting the *strategy-proofness* of φ at $(R_i, \bar{R}_j, m)$. Thus, $w = x$. Using a similar argument we go from $(R_i, \bar{R}_j, m)$ to (R_i, R_j, m), arriving at $y = w = x$. Thus, $\varphi^k(R_i, R_j, m) = x^k$, as desired.

Case 2. $x^k \notin S^k((r_i, r_j), m^k)$. As explained above, we need to prove that y^k is the closest point to x^k in $S^k((r_i, r_j), m^k)$. Now, $x^k \notin S^k((r_i, r_j), m^k)$ implies either $x_i^k > r_i$ or $x_j^k > r_j$. Without loss of generality, assume the former. Then, the closest point to x^k in $S^k((r_i, r_j), m^k)$ is $z \in S^k((r_i, r_j), m^k)$ such that $z_i = r_i$ and $z_j = m^k - z_i$.

By way of contradiction, suppose that $y \neq z$. Since by assumption $r_i + r_j \geq m^k$, by *same-sidedness*, $y_i^k < r_i = z_i$. Let $R_i' \in \mathcal{R}_i$ be such that $p(R_i') = p(R_i)$ and $x_i \; P_i' \; y_i$. Let $w \equiv \varphi(R_i', R_j, m)$. By Lemma 10, φ is *non-bossy* and thus, by Lemma 7, φ is *separable*. Clearly, *separability* implies *peaks-onliness*. Thus, $w = y$. Note that $r_i + r_j \geq m^k = x_i^k + x_j^k$ and $p_i \geq x_i^k > r_i$ implies $r_j > x_j^k$. Thus, $x^k \in S^k((p_i, r_j), m^k)$. Thus, by Case 1, $x^k = \varphi^k(\bar{R}_i, R_j, m)$. Since, as noted in the beginning of the proof, φ is *separable*, for each $l \in K \setminus \{k\}$, $x^l = \varphi^l(\bar{R}_i, R_j, m) = w^l$. Thus,

$$\varphi_i(\bar{R}_i, R_j, m) = x_i \; P_i' \; w_i = \varphi_i(R_i', R_j, m),$$

contradicting *strategy-proofness* at (R_i', R_j, m). Thus, $y^k = z$, establishing (15).

In the case where $r_i + r_j \leq m^k$, a symmetric argument, using (ii) above instead of (i), again establishes $\varphi^k(R_i, R_j, m) = \phi^k(R_i, R_j, m; u)$. □

Lemma 12. *Let φ denote a mechanism satisfying strategy-proofness, consistency, physical resource-monotonicity, and same-sidedness.*

Then, there is u in $\mathcal{U}$ such that, for each $N \in \mathcal{N}$ and each $(R, m) \in \mathcal{E}^N$, $\varphi(R, m) = \phi(R, m; u)$.

Proof. Let φ denote a mechanism satisfying the properties Lemma 12. Let $N \in \mathcal{N}$, $(R, m) \in \mathcal{E}^N$, and $x \equiv \varphi(R, m)$. By the *consistency* of φ, for each $\{i, j\} \subseteq N$, $x_{\{i,j\}} = \varphi(R_i, R_j, x_i + x_j)$. By Lemma 11, there is a u in $\mathcal{U}$ such that, for each $\{i, j\} \subseteq N$, $\varphi(R_i, R_j, x_i + x_j) = \phi(R_i, R_j, x_i + x_j; u)$. By Lemma 2, $\phi(\cdot; u)$ is *conversely consistent*. Thus, $\phi(R, m; u) = x$. Thus, $\phi(R, m; u) = \varphi(R, m)$. □

Proof of Theorem 1. By Lemma 2, each separably concave mechanism is *strategy-proof*, *unanimous*, *consistent*, and *resource-monotonic*. Conversely, let φ denote a *strategy-proof*, *unanimous*, *consistent*, and *resource-monotonic* mechanism. By Lemma 10, φ is *non-bossy*. Thus, by Lemma 3, φ is *same-sided*. Thus, by Lemma 4, φ is *peaks-only*. Thus, by Lemma 5, φ is *physically resource-monotonic*. Thus, by Lemma 12, there is a u in $\mathcal{U}$ such that, for each $N \in \mathcal{N}$ and each $(R, m) \in \mathcal{E}^N$, $\varphi(R, m) = \phi(R, m; u)$. Thus, φ is a separably concave mechanism. □

Proof of Theorem 2. By Lemma 2, each separably concave mechanism is *strategy-proof*, *unanimous*, *consistent*, and *physically resource-monotonic*. Conversely, let φ denote a *strategy-proof*, *unanimous*, *consistent*, and *physically resource-monotonic* mechanism. By Lemma 10, φ is *non-bossy*. Thus, by Lemma 3, φ is *same-sided*. Thus, by Lemma 12, there is a u in $\mathcal{U}$ such that, for each $N \in \mathcal{N}$ and each $(R, m) \in \mathcal{E}^N$, $\varphi(R, m) = \phi(R, m; u)$. Thus, φ is a separably concave mechanism. □

A.4. Results for the single resource case

We first recall useful facts (Sprumont, 1991) that help prove Proposition 2.

Remark 2. Suppose that K is a singleton. Let $N \in \mathcal{N}$ and $(R, m) \in \mathcal{E}^N$.

(i) $P(R, m)$ is a compact and convex set.
(ii) Allocation $x \in Z(N, m)$ is *efficient* at $(R, m) \in \mathcal{E}^N$ if and only if $\sum_N p(R_i) \geq m$ implies $x \leq p(R)$, and $\sum_N p(R_i) \leq m$ implies $x \geq p(R)$.

Proof of Proposition 2. Suppose that K is a singleton. Let φ denote a *strategy-proof*, *unanimous*, and *consistent* mechanism. We will prove that φ is *efficient*. If φ is not *efficient*, there are $N \in \mathcal{N}$ and $(R, m) \in \mathcal{E}^N$ such that $x \equiv \varphi(R, m)$ is not *efficient* at (R, m), $x \notin P(R, m)$. Thus, by (ii) in Remark 2, there is a pair $i, j \in N$ such that $x_i < p(R_i)$ and $x_j > p(R_j)$. By *consistency*, $x_{\{i,j\}} = \varphi(R_i, R_j, x_i + x_j)$. By *unanimity*, either $x_i + x_j < p(R_i) + p(R_j)$ or $x_i + x_j > p(R_i) + p(R_j)$. Without loss of generality, assume the former. Let $R'_i \in \mathcal{R}_i$ be such that $p(R'_i) = x_i + x_j - p(R_j) > 0$ and $y \equiv \varphi(R'_i, R_j, x_i + x_j)$. Since, $p(R'_i) + p(R_j) = x_i + x_j$, by *unanimity*, $y_i = p(R'_i)$ and $y_j = p(R_j)$. However, by feasibility, $x_i < y_i < p(R_i)$. Thus, $y_i \; P_i \; x_i$, contradicting *strategy-proofness* at (R_i, R_j). This contradiction establishes that φ is *efficient*. □

Finally we establish that the separably concave mechanisms are *group strategy-proof* when K is a singleton.

Proof of Proposition 1. Let $N \in \mathcal{N}$, $u \equiv \{(u_i, v_i) : i \in A\} \in \mathcal{U}$, $(R, m) \in \mathcal{E}^N$, and $x \equiv \phi(R, m; u)$. If $\sum_N p(R_i) = m$, then, for each $i \in N$, $x_i = p(R_i)$ and no agent has an incentive to misreport her preferences. Suppose that $\sum_N p(R_i) > m$.

Let $M \subseteq N$ and $(R', m) \in \mathcal{E}^N$ be such that, for each $j \in N \setminus M$, $R'_j = R_j$. Let $x' \equiv \phi(R', m; u)$ and assume that

$$\text{for each } i \in M,\; x'_i \; R_i \; x_i. \tag{16}$$

We will prove that (16) implies $x'_M = x_M$. By Proposition 2, $\phi(\cdot\,; u)$ is *efficient*. Thus, by Remark 2, $x \leq p(R)$. Because preferences are single-peaked, a necessary condition for (16) is that

$$\text{for each } i \in M,\; x_i \leq x'_i. \tag{17}$$

Case 1. $\sum_N p(R'_j) \leq m$. By Proposition 2, $\phi(\cdot\,; u)$ is *efficient*. Thus, by Remark 2, for each $j \in N \setminus M$, $x'_j \geq p(R'_j) = p(R_j) \geq x_j$. Thus, $\sum_M x_k \geq \sum_M x'_k$. But by (17), $\sum_M x_k \leq \sum_M x'_k$. Thus, $\sum_M x_k = \sum_M x'_k$. Thus, by (17), for each $j \in M$, $x'_j = x_j$, and $x'_M = x_M$, as desired.

Case 2. $\sum_N p(R'_j) > m$ and $x \in P(R', m)$. Thus, by the definition of $\phi(\cdot\,; u)$, $x' \neq x$ requires $x' \in P(R', m) \setminus P(R, m)$. Then, by Remark 2, there is $i \in N$ such that $p(R'_i) \geq x'_i > p(R_i) \geq x_i$. Thus, $P(R'_i, R_{N\setminus\{i\}}, m) \supseteq P(R, m)$ and, since $R_{N\setminus M} = R'_{N\setminus M}$, $i \in M$. Additionally, because preferences are single-peaked, (16) and $x'_i > p(R_i) \geq x_i$ requires $p(R_i) > x_i$. Thus, from the definition of $\phi(\cdot\,; u)$ and by Remark 2, $x = \phi(R'_i, R_{N\setminus\{i\}}, m; u)$. Thus, $x' \in P(R', m) \setminus P(R'_i, R_{N\setminus\{i\}}, m)$. Thus, there is $j \in N \setminus \{i\}$ such that $p(R'_j) \geq x'_j > p(R_j) \geq x_j$. Thus, $P(R'_i, R'_j, R_{N\setminus\{i\}}, m) \supseteq P(R, m)$ and, since $R_{N\setminus M} = R'_{N\setminus M}$, $j \in M$. Additionally, because preferences are single-peaked, (16) and $x'_j > p(R_j) \geq x_j$ requires $p(R_j) > x_j$. Thus, from the definition of $\phi(\cdot\,; u)$ and by Remark 2, $x = \phi(R'_i, R'_j, R_{N\setminus\{i,j\}}, m; u)$. Clearly, we can continue in this way until we exhaust M. Thus, (17) implies $x = x'$, and $x'_M = x_M$, as desired.

Case 3. $\sum_N p(R'_j) > m$ and $x \notin P(R', m)$. Then, by Remark 2, there is $i \in N$ such that $p(R_i) \geq x_i > p(R'_i) \geq x'_i$. Since $R_{N\setminus M} = R'_{N\setminus M}$, $i \in M$. This contradicts (17). Thus, (17) implies $x = x'$, and $x'_M = x_M$, as desired.

Cases 1 through 3 establish that, when $\sum_N p(R_i) \geq m$, (16) implies $x'_M = x_M$. A symmetric argument yields the same conclusion when $\sum_N p(R_i) \leq m$. Clearly, these conclusions hold for each $(R, m) \in \mathcal{E}^N$ and each group of agents $M \subseteq N$. Thus, $\phi(\cdot\,; u)$ is *group strategy-proof*. □

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