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Multi-item Vickrey–English–Dutch auctions ☆

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ABSTRACT

Assuming that bidders wish to acquire at most one item, this paper defines a polynomial time multi-item auction that locates the VCG prices in a finite number of iterations for any given starting prices. This auction is called the Vickrey–English–Dutch auction and it contains the Vickrey–English auction [Sankaran, J.K., 1994. On a dynamic auction mechanism for a bilateral assignment problem. Math. Soc. Sci. 28, 143–150] and the Vickrey–Dutch auction [Mishra, D., Parkes, D., 2009. Multi-item Vickrey–Dutch auctions. Games Econ. Behav. 66, 326–347] as special cases. By means of numerical experiments, it is showed that when the auctioneer knows the bidders' value distributions, the Vickrey–English–Dutch auction is weakly faster than the Vickrey–English auction and the Vickrey–Dutch auction in 89 percent and 99 percent, respectively, of the investigated problems. A greedy version of the Vickrey–English–Dutch auction is demonstrated to perform even better in the simulation studies.

1. Introduction

A fundamental insight due to Vickrey (1961) is that any sealed-bid auction mechanism that implements the unique minimum Walrasian equilibrium prices, often referred to as the Vickrey–Clarke–Groves prices (VCG prices for short), satisfies several desirable properties whenever the items are homogeneous and each bidder wish to acquire at most one item. For example, no bidder can gain by strategic misrepresentation, and the auction generates an efficient and individual rational outcome. Demange and Gale (1985) and Leonard (1983) demonstrated that the properties of this sealed-bid auction mechanism hold also when the items are heterogeneous (but still assuming unit-demand bidders).[1] Even if this sealed-bid auction satisfies many desirable properties, it is well-known that bidders often prefer iterative auction mechanisms (Engelberecht-Wiggans and Kahn, 1991; Cramton, 1998). There are many reasons for this. For example, an iterative format does not imply full preference revelation and it creates more transparency in the auctioneer's methods. This insight has motivated a substantial amount of research.

In an early paper, Demange et al. (1986) described an English multi-item auction for heterogeneous items based on the Hungarian method of Kuhn (1955). In this iterative auction format, prices are updated based on information regarding groups of items that are overdemanded. Here, a set of items is overdemanded, at a given price vector, if the number

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[1] A more demanding problem, not considered in this paper, is when bidders' are allowed to demand multiple items. See e.g. Ausubel (2004, 2006), Ausubel and Milgrom (2002), Bikhchandani et al. (2011), Bikhchandani and Ostroy (2002), Conen and Sandholm (2002), Cramton et al. (2006), de Vries et al. (2007), Gul and Staccetti (2000), Mishra and Parkes (2007), and Perry and Reny (2005).

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of bidders demanding only items in the set is greater than the number of items in the set. This is a natural approach since a necessary requirement for reaching the VCG prices is that all overdemanded sets of items are eliminated (Hall, 1935).[2] Even if the iterative auction mechanism in Demange et al. (1986) converges to the VCG prices in a finite number of iterations,[3] the price path from the sellers reservation prices to the VCG prices is not unique as it depends on the specific selection of the overdemanded set of items whose prices are updated for the next iteration. In fact, there are typically a very large number of different paths and it is ex ante not possible to identify the fastest (Andersson and Andersson, 2012).[4] Another fundamental problem with the approach in Demange et al. (1986) and related papers (Andersson et al., 2010; Sun and Yang, 2009, among others) is that the termination criteria require an exhaustive search of all subsets of items. This problem is clearly exponential which is a huge problem if the auction contains many items.

The above two problems with the multi-item auction in Demange et al. (1986) can be overcome by using a modification of the mechanism based on the Ford and Fulkerson (1956) algorithm as demonstrated by Andersson et al. (2010), Sankaran (1994), and Mo et al. (1988). This polynomial time unique path English multi-item auction is called the Vickrey–English auction (VE, henceforth) and it always converges to the VCG prices. VE starts at the sellers reservation prices. At these prices there are no weakly underdemanded sets of items (see footnote 2) by construction. The price increases in VE, prescribed by the Ford–Fulkerson algorithm, guarantee that the family of weakly underdemanded sets stays empty in the ascending process. Because all overdemanded sets of items always are eliminated after a finite number of price increases, as no item is infinitely valuable for any bidder, convergence to the VCG prices follows (again, see footnote 2).

The descending counterpart of the English format is the Dutch auction. Such a polynomial time unique path multi-item auction is defined in Mishra and Parkes (2009). This mechanism, called the Vickrey–Dutch auction (VD, henceforth), also identifies the VCG prices in a finite number of iterations. Convergence to the VCG prices follows by symmetrical arguments as in the above, i.e., VD starts at the upper bound of the price space where there are no overdemanded sets of items by construction. The prescribed price decreases then guarantee that the family of overdemanded sets of items stays empty in the descending process while the family of weakly underdemanded sets of items weakly shrinks in each iteration until it is empty.

The main innovation of this paper is the construction of a new polynomial time auction format called the Vickrey–English–Dutch auction (VED, henceforth). The fundamental difference between this mechanism compared to VE and VD is that it is allowed start at an arbitrary vector in the price space and yet locate the VCG prices, i.e., it need not start at the sellers reservation prices, as VE, or at the upper bound of the price space, as VD. Note, however, that VED can start at these prices as any price vector in the price space is an allowed starting point. In this case, VED is identical to VE and VD, and the two latter formats can therefore be regarded as special cases of VED. Note also that VED is not necessarily an ascending or descending format as both price increases and price decreases are allowed in the iterative process.

It is clear that any iterative auction format that converges to the VCG prices generates the same revenue independently of its starting price. Thus, all other things being equal, it is not unreasonable that the auctioneer selects the auction format which has the lowest expected number of iterations before convergence. One motivation for this is the fact that bidders' typically prefer auctions with a shorter running time over auctions with a long running time (Larson and Sandholm, 2001; Parkes et al., 1999). However, in order to evaluate the expected number of iterations, the auctioneer must have a measure of the number of required iterations before convergence when comparing the performance of different auction formats. This paper demonstrates that the measure of the number of iterations between any two price vectors on the path from the starting prices to the VCG prices, for VED, can be based on the Chebyshev metric.

Because VED gives the auctioneer the freedom to start at an arbitrary price vector in the price space, any information regarding the bidders' may help the auctioneer to reduce the expected number of iterations (this information may be the distribution of valuations, bidding behavior in previous auctions, etc.). This is easiest seen by considering the bounding cases of VED, namely VE and VD, and noting that VE (VD) is an ascending (descending) format. It is therefore impossible to decrease (increase) the prices at any iteration. This in built feature of these formats forces the auctioneer to start at the lowest (highest) possible price in the price space to guarantee convergence to the VCG prices. Hence, neither VE nor VD can take advantage of more detailed information about the bidders. For example, if the auctioneer have information about how the valuations of the bidders' are distributed, VE and VD must still start at the lowest and highest possible price in the price space, respectively. For an auctioneer that adopts VED with a flexible starting price, on the other hand, it is easy (e.g. by means of simulations) to find the expected VCG prices and then select this expectation as starting prices. This will obviously reduce the expected number of iterations as illustrated in this paper by means of numerical experiments. Note also that VED always converges to the VCG prices even if the auctioneer does not have any information about the bidders.

[2] A necessary and sufficient condition for a price vector to be a VCG price vector is that all overdemanded and all weakly underdemanded sets of items are eliminated (Mishra and Talman, 2010, Theorem 2). A set of items is weakly underdemanded, at a given price vector, if the number of bidders that demand some item in the set is weakly lower than the number of items in the set, and the price of each item in the set is strictly higher that the seller's reservation prices (see Definition 3).

[3] This result holds if bidders report truthfully in the iterative process. However, truthful bidding is not a dominant strategy even if the mechanism converges to the VCG prices. This is in sharp contrast to its sealed-bid counterpart. However, truthful revelation constitutes a Nash equilibrium given that certain "activity rules" are imposed. See, e.g., Ausubel (2006), de Vries et al. (2007), Gul and Staccetti (2000), or Parkes (2001) a for detailed discussion and analysis. See also Ausubel (2006).

[4] All possible paths from the sellers reservation prices to the VCG prices, for the family of English auctions with unit-demand bidders, are characterized in Andersson et al. (2010).

The main point here is that *if* the auctioneer has some information, it may be used cleverly when selecting the starting prices to reduce the expected number of iterations. This is neither possible for VE nor for VD.

As explained in the above, the Vickrey–English–Dutch auction always locates the VCG prices in polynomial time for any given starting prices. In the unit-item auction case, the Bisection auction (Grigorieva et al., 2007) shares this property. A more general model is considered by Ausubel (2006) where a number of heterogeneous items are to be allocated among a number of bidders that are allowed to demand multiple items. In this auction format, bidders report their demand sets at any price announced by the auctioneer. Given these reports, the auctioneer identifies the overdemanded and the underdemanded items and adjust prices up and down, respectively. This iterative process continues until a price vector is reached at which demand and supply are in balance for each item. For any starting prices, the process always converges to an equilibrium in finitely many iterations. The process of Ausubel (2006) is, however, fundamentally different from the Vickrey–English–Dutch auction, e.g., as payments in the latter is based only on the prices in the last step of the auction whereas payments in the former is based on the entire price path generated by the auction.

The paper is outlined as follows. Section 2 presents the basic model and the concept of a price equilibrium. Some elementary set definitions and a number of important properties are provided in Section 3. The Vickrey–English–Dutch auction is introduced in Section 4 where also some characteristics of it are provided. In addition, a greedy version of the Vickrey–English–Dutch auction is presented in Section 4.1. Section 5 contains some computational results obtained by numerical simulations. Section 6 concludes the paper.

2. Model and price equilibria

Let the finite sets of *items* and *bidders* be denoted by $I = \{1, \dots m\}$ and $B = \{1, \dots n\}$, respectively. Let also v_{bi} represent bidder b's *valuation* of item i. Valuations are assumed to be non-negative integers drawn from a distribution function f with support on $[r, u]$. Here, r represents the seller's *reservation price* vector, and it is assumed, without loss of generality, that $r = (0, \dots, 0)$. The vector u is the upper bound on valuations. Moreover, each bidder $b \in B$ knows her own valuation for every item in I, and all valuations are independent of each other.

There is a *null-item*, denoted by 0, which represents the situation when bidders are not assigned any item from I. The price and the valuation for the null-item is zero for all bidders, and it can be assigned to any number of bidders. For notational convenience, let $I^* = I \cup \{0\}$ and $I^+(p) = \{i \in I\colon\ p_i > 0\}$. Each bidder is interested in acquiring at most one item (unit-demand bidders). The prices of the items in I^* are gathered in the *price vector* $p = (p_0, p_1, \dots, p_m)$. The *net valuation* for bidder $b \in B$ of item $i \in I^*$ equals the valuation of the item minus its price, i.e., $v_{bi} - p_i$. Each bidder demands the items with the highest net valuation. Formally, the *demand correspondence* $D_b(p)$ for bidder $b \in B$ at price vector p is given by:

$$D_b(p) = \{i \in I^*\colon\ v_{bi} - p_i \geqslant v_{bj} - p_j \text{ for all } j \in I^*\}.$$

A price vector p is an *equilibrium price vector* if there is an assignment $x\colon B \to I^*$ such that $x_b \in D_b(p)$ for all $b \in B$, $x_b \neq x_{b'}$ if $x_b, x_{b'} \in I$ (with $b \neq b'$), and $p_i = 0$ if $x_b \neq i$ for all $b \in B$, i.e., each bidder is assigned an item from his demand set, each item different from the null-item can be assigned to at most one bidder and the price of any unassigned item equals the reservation price. The pair (x, p) is an *equilibrium allocation* if p is an equilibrium price vector.

For the above model, Shapley and Shubik (1972) demonstrated that the set of competitive price vectors is non-empty and forms a complete lattice. This result guarantees the existence of a unique minimal equilibrium price vector. We will refer to this unique vector as the VCG price vector (Vickrey, 1961; Clarke, 1971; Groves, 1973), and it will be denoted by p^{VCG}.

This section ends by introducing a few concepts and definitions from Mishra and Parkes (2009). A bidder $b \in B$ is *satisfied* at allocation (x, p) if $x_b \in D_b(p)$. Note that each bidder is satisfied at an equilibrium allocation. An assignment x is *admissible* at the price vector p if $x_b \in D_b(p) \cup \{0\}$ for all $b \in B$. An assignment x is *provisional* at the price vector p if it generates the maximum revenue among all admissible assignments, breaking ties in favor of satisfying the maximum number of bidders and then at random. Let $X(D(p))$ denote the set of provisional assignments at the price vector p, and let $X(D_{-b}(p))$ denote the set of provisional assignments at the price vector p when only considering bidders in $B_{-b} = B \setminus \{b\}$. Finally, let $A(x) \subseteq I^+(p)$ denote the set of assigned items in x with positive prices, i.e., $A(x) = \{i \in I^+(p)\colon\ x_j = i \text{ for some } j \in B\}$. Similarly, $A_{-b}(x) = \{i \in I^+(p)\colon\ x_j = i \text{ for some } j \in B \setminus \{b\}\}$.

Definition 1. Item $i \in I^*$ is *universally allocated* if $i \in I^* \setminus I^+(p)$ or if item $i \in I^+(p)$ is provisionally assigned to some bidder $b \in B$ and there exists a provisional assignment $y \in X(D_{-b}(p))$ where $A(x) = A_{-b}(y)$.

3. Set definitions and results

This section introduces a number of set definitions that all are based on the demand correspondences of the bidders. Two of these are of particular interest. Namely, the set in excess demand with maximal cardinality and the set in excess supply. Both these sets can be identified in polynomial time and will play a key role when defining the Vickrey–English–Dutch auction in the next section.

The set $O(S, p)$ contains the bidders $b \in B$ that only demand items in $S \subseteq I$ at price vector p, and the set $U(S, p)$ contains the bidders $b \in B$ that demand some item in $S \subseteq I$ at price vector p. Formally:

$$O(S,p) = \{b \in B\colon\ D_b(p) \subseteq S\},$$
$$U(S,p) = \{b \in B\colon\ D_b(p) \cap S \neq \emptyset\}.$$

Using $O(S,p)$ and $U(S,p)$, the central concepts of overdemanded and weakly underdemanded sets of items are next defined.

Definition 2. A set of items S is *overdemanded* at prices p if $S \subseteq I$ and $|O(S,p)| > |S|$.

Definition 3. A set of items S is *weakly underdemanded* at prices p if $S \subseteq I^+(p)$ and $|U(S,p)| \leqslant |S|$.

The family of overdemanded and weakly underdemanded sets of items, at prices p, are denoted by $OD(p)$ and $UD(p)$, respectively, i.e.:

$$OD(p) = \{S \subseteq I\colon\ |O(S,p)| > |S|\},$$
$$UD(p) = \{S \subseteq I^+(p)\colon\ |U(S,p)| \leqslant |S|\}.$$

Note that $OD(u) = \emptyset$, by construction, as $0 \in D_b(u)$ for all $b \in B$. Moreover, $UD(r) = \emptyset$ because no item i with $p_i = r_i$ can be weakly underdemanded by Definition 3. Next, the above definitions are illustrated in an example.

Example 1. Suppose that $B = \{1,2,3,4,5,6\}$, $I = \{1,2,3,4\}$, $p_i > r_i$ for $i = 1,4$, and $p_i = r_i$ for $i = 2,3$. Hence, $I^+(p) = \{1,4\}$. Furthermore, assume that $D_1(p) = D_2(p) = D_3(p) = \{1\}$, $D_4(p) = D_5(p) = \{2\}$, and $D_6(p) = \{4\}$. In this case, $OD(p) = \{\{1\},\{2\},\{1,2\},\{1,3\},\{1,4\},\{2,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},I\}$, and $UD(p) = \{4\}$.

The following result, which is due to Mishra and Talman (2010, Theorem 2), uses Definitions 2 and 3 to characterize the VCG price vector.

Theorem 1. *A price vector p equals the VCG price vector if and only if $OD(p) = \emptyset$ and $UD(p) = \emptyset$.*

Next, the concept of a set in excess demand is introduced. Informally, a set of items S is in excess demand, at given prices, if S is overdemanded and the number of items in each proper subset T of S is strictly smaller than the number of bidders that demand some item in T and in addition only demand items in S.[5]

Definition 4. A set of items S is in *excess demand* at prices p if $S \subseteq I$ and:

$$|U(T,p) \cap O(S,p)| > |T| \quad \text{for each non-empty } T \subseteq S. \tag{1}$$

Example 2. This example is based on Example 1. As any set in excess demand must be overdemanded, any set in excess demand must belong to $OD(p)$. However, only the sets $\{1\}$, $\{2\}$ and $\{1,2\}$ are in excess demand as condition (1) not is satisfied for the other sets in $OD(p)$. For example, if $S = \{2,4\}$ and $T = \{4\}$, it follows that $U(\{4\},p) = \{6\}$ and $O(\{2,4\},p) = \{4,5,6\}$. Consequently, $|U(T,p) \cap O(S,p)| = 1 = |T|$.

To define a stronger version of excess demand, a stronger version of the demand correspondence is needed. Let:

$$D_b^+(p) = D_b(p) \cap I^+(p),$$
$$O^+(S,p) = \{b \in B\colon\ D_b^+(p) \subseteq S\}.$$

Clearly, $D_b^+(p) \subseteq D_b(p)$. Using the above definitions, a set of items in positive excess demand is defined as follows.

Definition 5. A set of items S is in *positive excess demand* at prices p if $S \subseteq I^+(p)$ and:

$$|U(T,p) \cap O^+(S,p)| > |T| \quad \text{for each non-empty } T \subseteq S. \tag{2}$$

Example 3. This example is based on the same premises as Example 1. Because $I^+(p) = \{1,4\}$, it is clear that $D_1^+(p) = D_2^+(p) = D_3^+(p) = \{1\}$, $D_4^+(p) = D_5^+(p) = \emptyset$, and $D_6^+(p) = \{4\}$. Consequently, $O^+(\{1\},p) = \{1,2,3\}$, $O^+(\{4\},p) = \{6\}$, and $O^+(\{1,4\},p) = \{1,2,3,6\}$. Note next that a set in positive excess demand must be a subset of $I^+(p) = \{1,4\}$. However, $S = \{1\}$ is the only subset of $I^+(p)$ satisfying condition (2). Therefore, the set $S = \{1\}$ is the only set in positive excess demand at prices p.

[5] A detailed analysis of the family of sets in excess demand can be found in Andersson et al. (2010). Also, the definition of a set in excess demand coincides with the definition of a *pure overdemanded* set (Mo et al., 1988; Sankaran, 1994).

Theorem 2. *There exists a unique (possibly empty) set in positive excess demand (called $\tilde{S}$, henceforth) with maximal cardinality at any given price vector p. This set is identical to the set of all universally allocated items with positive prices (called $\hat{S}$, henceforth).*

Proof. See Appendix A. □

Next the concept of excess supply is introduced. This concept can be regarded as the supply counterpart of excess demand.

Definition 6. The set $S^* = I^+(p) \setminus \tilde{S}$ is in *excess supply* at prices p if $\tilde{S}$ is the set in positive excess demand with maximal cardinality at prices p.

Proposition 1. *For any price vector $p \geqslant r$:*

(i) *There exists a unique (possibly empty) set in excess demand with maximal cardinality E^*. This set can be identified in polynomial time using the Ford and Fulkerson (1956) algorithm.*
(ii) *There exists a unique (possibly empty) set in excess supply S^*. This set can be identified in polynomial time using the* FindUnivAllocItems *procedure (Mishra and Parkes, 2009).*

Proof. Part (i). The existence of a unique set in excess demand with maximal cardinality E^* is established in Andersson et al. (2010).[6] This set is the outcome of the Ford–Fulkerson algorithm (Andersson et al., 2010; Sankaran, 1994).

Part (ii). The set $\tilde{S}$ is unique by Theorem 2, and it can be identified in polynomial time using the FindUnivAllocItems procedure (see Mishra and Parkes, 2009, Proposition 2). Consequently, $S^* = I^+(p) \setminus \tilde{S}$ is unique, and can be found in polynomial time. □

Remark 1. The implication from Theorem 2 and the proof of Proposition 1(ii) is that the outcome of the FindUnivAllocItems procedure is the unique set in positive excess demand with maximal cardinality. In this sense, Theorem 2 characterizes the outcome of the FindUnivAllocItems procedure.

Example 4. This example is a continuation of Examples 1–3. Note first that, it is clear from Example 2 that $E^* = \{1, 2\}$ is the unique set in excess demand with maximal cardinality. Furthermore, from Example 3, we know that $\{1\}$ is the unique set in positive excess demand $\tilde{S}$ (this set is identical to the set of all universally allocated items with positive prices $\hat{S}$ by Theorem 2). Because $I^+(p) = \{1, 4\}$, it follows from Definition 6 that the unique set in excess supply is given by $S^* = \{1, 4\} \setminus \{1\} = \{4\}$.

4. The Vickrey–English–Dutch auction

This section introduces the Vickrey–English–Dutch auction (VED, henceforth) and states a few of its properties. We start by defining the price adjustments for VED. These adjustments are either based on the maximal set in excess demand or the set in excess supply. Let now t and $t+1$ be any two succeeding iterations. The difference between the price vectors p^{t+1} and p^t can formally be described as:

$$p_i^{t+1} = \begin{cases} p_i^t + \alpha & \text{if } i \in K_t, \\ p_i^t & \text{otherwise,} \end{cases}$$

where $K_t = E_t^*$ and $\alpha = 1$, or $K_t = S_t^*$ and $\alpha = -1$.[7] The former price adjustment is called an E^*-increase, and the latter is called an S^*-decrease. Obviously, an E^*-increase (S^*-decrease) is only possible if the set E_t^* (the set S_t^*) is non-empty. The following lemma establishes two monotonicity properties of these price adjustments.

Lemma 1. *For any price vector $p \geqslant r$:*

(i) *If $S_t^* = \emptyset$ and the price adjustment is given by an E^*-increase, then $S_{t+1}^* = \emptyset$.*
(ii) *If $E_t^* = \emptyset$ and the price adjustment is given by an S^*-decrease, then $E_{t+1}^* = \emptyset$.*

[6] From Definition 4, it is clear that a set in excess demand cannot be empty. However, to simplify notation, we define a set in excess demand to be empty whenever the family of sets in excess demand is empty or, equivalently, when the output from the Ford–Fulkerson algorithm is the empty set.

[7] E_t^* denotes the unique set in excess demand with maximal cardinality at iteration t, and S_t^* denotes the unique set in excess supply at iteration t.

Proof. Part (i) can be found in Andersson et al. (2010, Lemma 1). To prove part (ii), note that $E^* = \emptyset$ at prices q if and only if $OD(q) = \emptyset$. Consequently, the statement is true if $OD(p^{t+1}) = \emptyset$, i.e., if:

$$\left|O\left(T, p^{t+1}\right)\right| \leqslant |T| \quad \text{for all } T \subseteq I.$$

Let now T be any non-empty subset of I, and let $T = A \cup C$ where $A \subseteq S_t^*$ and $C \subseteq I \setminus S_t^*$. Note that $A \cap C = \emptyset$ by construction, and that A and/or C are non-empty as $T \neq \emptyset$.

Let now b be any bidder in $O(T, p^{t+1})$, and note that this means that $D_b(p^{t+1}) \cap A \neq \emptyset$ and/or $D_b(p^{t+1}) \cap C \neq \emptyset$. This leads to two observations:

- If $D_b(p^{t+1}) \cap C \neq \emptyset$, then $b \in O(C, p^t)$. To see this, suppose that $l \in D_b(p^t)$ for some $l \notin C$. Now if $l \in S_t^*$, then $C \cap D_b(p^{t+1}) = \emptyset$ as:

$$v_{bl} - p_l^{t+1} > v_{bl} - p_l^t \geqslant v_{bk} - p_k^t = v_{bk} - p_k^{t+1} \quad \text{for all } k \in C,$$

  which is a contradiction to $D_b(p^{t+1}) \cap C \neq \emptyset$. If $l \in I \setminus (S_t^* \cup C)$, then $l \in D_b(p^{t+1})$ as $D_b(p^{t+1}) \cap C \neq \emptyset$, which contradicts that $b \in O(T, p^{t+1})$.
- If $D_b(p^{t+1}) \cap A \neq \emptyset$, then $b \in U(A, p^t)$ or $b \in O(C, p^t)$. To see this, suppose that $b \notin U(A, p^t)$. Because the demand correspondence always is non-empty, there must be an item $l \in D_b(p^t)$. Moreover, as $b \notin U(A, p^t)$ it is clear that $v_{bl} - p_l^t > v_{bk} - p_k^t$ for all $k \in A$. But as $D_b(p^{t+1}) \cap A \neq \emptyset$, it must then be the case that $v_{bk'} - p_{k'}^{t+1} = v_{bl} - p_l^t$ for some $k' \in A$. Hence, $l \notin S_t^*$ and $l \in D_b(p^{t+1})$. But then $l \in C$ as $b \in O(T, p^{t+1})$. Hence, $D_b(p^{t+1}) \cap C \neq \emptyset$, and consequently, $b \in O(C, p^t)$ by the above conclusion.

In summary, if $b \in O(T, p^{t+1})$ then $b \in O(C, p^t) \cup U(A, p^t)$. Hence, $O(T, p^{t+1}) \subseteq (O(C, p^t) \cup U(A, p^t))$. This together with Definitions 2–3 and $A \cap C = \emptyset$ gives:

$$\left|O\left(T, p^{t+1}\right)\right| \leqslant \left|\left(O\left(C, p^t\right) \cup U\left(A, p^t\right)\right)\right| = \left|O\left(C, p^t\right)\right| + \left|U\left(A, p^t\right)\right|. \tag{3}$$

Note next that $|O(C, p^t)| \leqslant |C|$ as $OD(p^t) = \emptyset$. Note also that $|U(A, p^t)| \leqslant A$ since no item in A is universally allocated. These facts together with condition (3) give the desired conclusion. □

Using the insights from Lemma 1 in combination with Theorem 1, it may be tempting to define an iterative auction that is allowed to start at an arbitrary price vector p^s which, in each iteration t, increases the price of the items in E_t^* (if non-empty) *and* decreases the price of the items in S_t^* (if non-empty). However, even if this iterative process appears to be promising, such an iterative auction may run into difficulties as illustrated in the following example.

Example 5. Suppose that $I = I_1 \cup I_2$ where $|B| > |I|$, $I_1 \cap I_2 = \emptyset$ and $|I_i| \geqslant 1$ for $i = 1, 2$. Consider valuations such that $v_{bi} - p_i^s - (v_{bj} - p_j^s) = \Delta$ for all $i \in I_1$, all $j \in I_2$ and all $b \in B$. The difference Δ is an odd integer. Now, if the auction starts at price vector p^s and E^*-increases *and* S^*-decreases are performed whenever possible, then:

- $E_t^* = I_1$ and $S_t^* = I_2$ for all iterations $1 \leqslant t \leqslant 0.5(\Delta + 1)$,
- $E_t^* = I_2$ and $S_t^* = I_1$ for iteration $t = 0.5(\Delta + 1) + k$,
- $E_t^* = I_1$ and $S_t^* = I_2$ for iteration $t = 0.5(\Delta + 1) + k + 1$,

where $k \geqslant 1$ is an odd integer. This demonstrates that the above iterative auction is trapped in a cycle for any iteration $t > 0.5(\Delta + 1)$.[8]

The insight from Example 5 is that the iterative process may get trapped into a cycle[9] if the auctioneer is too greedy in finding a short path from the starting prices to the VCG prices, i.e., if prices are adjusted for the items in E_t^* *and* S_t^* simultaneously. The primary solution to this problem, proposed in this paper, is to only adjust the prices in E_t^* *or* S_t^* (see also the discussion in Section 4.1).

[8] Here is a short numerical example which is based on the same premisses as Example 5. Suppose that $I = \{1, 2\}$, $I_1 = \{1\}$, $I_2 = \{2\}$, $B = \{1, 2, 3\}$, $v_{i1} = 9$ and $v_{i2} = 2$ for all bidders. The starting prices is 5 for both items, i.e. $p^s = (5, 5)$. In the first four iterations, $E_t^* = \{1\}$ and $S_t^* = \{2\}$. Hence, the price for item 1 increases from 5 to 9, and the price for item 2 decreases from 5 to 1. At iteration 5, where prices are given by $(p_1, p_2) = (9, 1)$, item 2 is overdemanded and item 1 is weakly underdemanded, i.e., $E_5^* = \{2\}$, and $S_5^* = \{1\}$. Hence, the price of item 1 decreases to 8 and the price of item 2 increases to 2. But then, we are back at the price vector $(8, 2)$ at iteration 6. Consequently, the prices jump back to $(9, 1)$ in iteration 7. Hence, the iterative process will from now on jump back and forth between price vectors $(8, 2)$ and $(9, 1)$.

[9] Of course, it is possible to define rules that force the process to leave a cycle but there is no guarantee that such rules will prevent the algorithm to end up in another cycle in later iterations. For a simple rule that avoids further cycling, see Section 4.1.

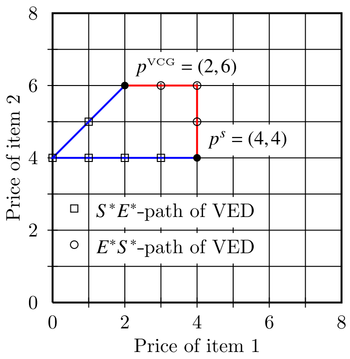

Fig. 1. The E^*S^*-path and the S^*E^*-path of the Vickrey–English–Dutch auction for starting prices $p^s = (4,4)$.

Algorithm 1 *(Vickrey–English–Dutch multi-item auction).* Initialize the price vector to the starting prices $p^s \in \mathbb{N}^{m+1}$. For each iteration $t = 0, 1, \ldots$:

1. Collect the demand sets $D_b(p^t)$ of every bidder $b \in B$.
2. If $E_t^* = S_t^* = \emptyset$ at p^t, terminate the algorithm. Otherwise, goto Step 3.
3. Identify the set E_t^*. If $E_t^* = \emptyset$, goto Step 4. Otherwise, let p^{t+1} be given by an E^*-increase. Set $t := t+1$ and start a new iteration from Step 1.
4. Identify the set S_t^*. If $S_t^* = \emptyset$, goto Step 2. Otherwise, let p^{t+1} be given by an S^*-decrease. Set $t := t+1$ and start a new iteration from Step 1.

Remark 2. Note that VED eliminates all overdemanded sets of items (Step 3) before eliminating all weakly underdemanded sets of items (Step 4). This generates a unique sequence of price vectors $(p^s, \ldots, p^{\text{VCG}})$ called the E^*S^*-path. Alternatively, Steps 3 and 4 can be reversed, i.e., the auctioneer eliminates all overdemanded sets after all weakly underdemanded sets have been eliminated. Also this generates a unique path, referred to as the S^*E^*-path, henceforth.

Remark 3. It is clear that the Vickrey–English auction (Sankaran, 1994, Algorithm DGS′ together with Step 3′) and the Vickrey–Dutch auction (Mishra and Parkes, 2009, Definition 5) are special cases of VED. This follows as the sets E_t^* and S_t^*, used in Algorithm 1, characterize the outcome of the Ford–Fulkerson algorithm and the FindUnivAllocItems procedure, respectively, by Proposition 1. Note also that the Vickrey–English auction (Vickrey–Dutch auction) assumes that $p^s = r$ ($p^s = u$) and performs an E^*-increase (S^*-decrease) in each iteration until the set E_t^* (the set S_t^*) is empty. Because $S_t^* = \emptyset$ at prices r ($E_t^* = \emptyset$ at prices u), by definition, Lemma 1 guarantees that the set S_t^* (the set E_t^*) stays empty in the iterative process. Hence, Step 4 (Step 3) of Algorithm 1 is redundant in the Vickrey–English auction (Vickrey–Dutch auction).

Theorem 3. *Algorithm 1 converges to the VCG prices in a finite number of iterations.*

Proof. Consider any starting prices p^s in the price space. The auctioneer can choose either the E^*S^*-path or the S^*E^*-path of Algorithm 1. Suppose that the E^*S^*-path (S^*E^*-path) is selected. As soon as the prices have increased (decreased) sufficiently much, there cannot be any overdemanded (weakly underdemanded) sets of items as no item is infinitely good or bad for any bidder. Hence, $E_t^* = \emptyset$ ($S_t^* = \emptyset$) after finitely many iterations. Lemma 1(ii) (Lemma 1(i)) guarantees that E_t^* (S_t^*) stays empty when the S^*-decreases (E^*-increases) are conducted. Again, as no item is infinitely good or bad for any bidder, the set S_t^* (the set E_t^*) will be empty after a finite number of iterations. But if $E_t^* = S_t^* = \emptyset$, Theorem 1 gives the desired conclusion. □

Example 6. Let $B = \{a, b, c\}$, $I = \{1, 2\}$, $u = (8, 8)$, $r = (0, 0)$, and:

$$v = \begin{pmatrix} v_{a1} & v_{a2} \\ v_{b1} & v_{b2} \\ v_{c1} & v_{c2} \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 3 & 7 \\ 6 & 7 \end{pmatrix}.$$

Here, the VCG prices are given by $p^{\text{VCG}} = (2, 6)$. For this example, the starting prices are set to $p^s = (4, 4)$. If the E^*S^*-path of VED is adopted, $E_0^* = E_1^* = \{2\}$ and $E_2^* = \emptyset$, i.e., $E_2^* = \emptyset$ at prices $(4, 6)$. From this coordinate, only S^*-increases are performed. More specifically, $S_2^* = S_3^* = \{1\}$ and $S_4^* = \emptyset$ $(= E_2^* = E_3^* = E_4^*)$. In total, four iterations are required before converge to the VCG prices. If, on the other hand, the S^*E^*-path of VED is adopted, $S_0^* = S_1^* = S_2^* = S_3^* = \{1\}$ and $S_4^* = \emptyset$. Hence, from coordinate $(0, 4)$, only D^*-increases are conducted. More specifically, $E_4 = E_5 = \{2\}$ and $E_6^* = \emptyset$ $(= S_4^* = S_5^* = S_6^*)$, meaning that six iterations in total are required before termination. See Fig. 1 for an illustration.

As demonstrated in Theorem 4, below, the following two definitions will play a key role when determining the number of required iterations for VED before convergence to the VCG prices.

Definition 7. A sequence of price vectors $(p^0, \dots, p^n)$ is called a *path* if $\max_{i \in I}(|p_i^t - p_i^{t+1}|) = 1$ for all $0 \leqslant t \leqslant n-1$.

Definition 8. Let $(d_\infty, \mathbb{N}^{m+1})$ be a metric space where d_∞ is the Chebyshev metric (or Tchebychev metric or Maximum metric). Formally:

$$d_\infty(p, q) = \max_i\bigl(|p_i - q_i|\bigr),$$

where p and q are two price vectors in $\mathbb{N}^{m+1}$.

Note that a sequence of price vectors $(p^0, \dots, p^n)$ generated by E^*-increases (S^*-decreases) always constitutes a path as the price adjustment for each item $i \in I$ is at most one (minus one) when comparing two succeeding price vectors p^t and p^{t+1} in the sequence.

Theorem 4. *Consider a path $(p^0, \dots, p^n)$ produced from E^*-increases (S^*-decreases) such that $p^s = p^0$, and p^n is the first price vector in the path where there are no non-empty sets in excess demand (excess supply). Then $d_\infty(p^0, p^n)$ gives the number of E^*-increases (S^*-decreases) from p^0 to p^n.*

Proof. We only prove the result for E^*-increases as the proof is symmetrical for S^*-decreases. Suppose first that $d_\infty(p^0, p^n) = 0$, which means that $p^0 = p^n$. Since, there cannot be any cycles when E^*-increases are conducted, there cannot be any overdemanded sets of items at $p^n = p^0$. Thus, the number of E^*-increases equals zero.

Suppose instead that $d_\infty(p^0, p^n) = 1$. Let E^* denote the set of items with increased prices, i.e. $p_i^n = p_i^0 + 1$ for all $i \in E^*$. Decompose the set E^* into $n-1$ non-empty subsets $E_0^*, \dots, E_{n-1}^*$, where subset E_i^* is the unique non-empty set in excess demand with maximal cardinality at prices p^i (note that at prices p^n, there are no non-empty sets in excess demand by definition). Hence, $E^* = E_0^* \cup \dots \cup E_{n-1}^*$. Note that $E_j^* \cap E_k^* = \emptyset$ for all $j \neq k$ as $d_\infty(p^0, p^n) = 1$. To obtain a contradiction, suppose that $n \geqslant 2$. We will demonstrate that the set E^* is in excess demand at prices p^0, which contradicts the assumption that E_0^* is the set in excess demand with maximal cardinality at prices p^0.

Let G be an arbitrary non-empty subset of E^*. Hence, $G \subseteq E_0^* \cup \dots \cup E_{n-1}^*$. Let $G_i = G \cap E_i^*$ for all $i = 0, \dots, n-1$. Then G may be rewritten as:

$$G = \bigl(G \cap E_0^*\bigr) \cup \dots \cup \bigl(G \cap E_{n-1}^*\bigr) = G_1 \cup \dots \cup G_{n-1}.$$

This partition, the distributive law of set theory, and the facts that $O(E_i^*, p^0) \subseteq O(E^*, p^0)$ for all $i = 0, \dots, n-1$, and $O(E_j^*, p^0) \cap O(E_k^*, p^0) = \emptyset$ for all $j \neq k$ yield:

$$\begin{aligned}
\bigl|O\bigl(E^*, p^0\bigr) \cap U\bigl(G, p^0\bigr)\bigr| &= \bigl|O\bigl(E^*, p^0\bigr) \cap \bigl(U\bigl(G_0, p^0\bigr) \cup \dots \cup U\bigl(G_{n-1}, p^0\bigr)\bigr)\bigr|, && (4)\\
&= \bigl|\bigl(O\bigl(E^*, p^0\bigr) \cap U\bigl(G_0, p^0\bigr)\bigr) \cup \dots \cup \bigl(O\bigl(E^*, p^0\bigr) \cap U\bigl(G_{n-1}, p^0\bigr)\bigr)\bigr|, && (5)\\
&\geqslant \bigl|\bigl(O\bigl(E_0^*, p^0\bigr) \cap U\bigl(G_0, p^0\bigr)\bigr) \cup \dots \cup \bigl(O\bigl(E_{n-1}^*, p^0\bigr) \cap U\bigl(G_{n-1}, p^0\bigr)\bigr)\bigr|, && (6)\\
&= \bigl|\bigl(O\bigl(E_0^*, p^0\bigr) \cap U\bigl(G_0, p^0\bigr)\bigr)\bigr| + \dots + \bigl|\bigl(O\bigl(E_{n-1}^*, p^0\bigr) \cap U\bigl(G_{n-1}, p^0\bigr)\bigr)\bigr|. && (7)
\end{aligned}$$

Note next that $O(E_i^*, p^i) \subseteq O(E_i^*, p^0)$ for all $i = 0, \dots, n$, and that $U(G_i, p^i) \subseteq U(G_i, p^0)$ for all $i = 1 \dots, n$. Thus, the following inequality holds:

$$\bigl|O\bigl(E_i^*, p^0\bigr) \cap U\bigl(G_i, p^0\bigr)\bigr| \geqslant \bigl|O\bigl(E_i^*, p^i\bigr) \cap U\bigl(G_i, p^i\bigr)\bigr|. \tag{8}$$

Recall next that $G_i \subseteq E_i^*$, and that E_i^* is the maximal set in excess demand at prices p^i. From Definition 4, it then follows:

$$\bigl|O\bigl(E_i^*, p^i\bigr) \cap U\bigl(G_i, p^i\bigr)\bigr| > |G_i| \quad \text{for all } i = 0, \dots, n-1. \tag{9}$$

Recalling that $G_j \cap G_k = \emptyset$ and $G = G_0 \cup \dots \cup G_{n-1}$, by construction, and using conditions (4)–(9), we obtain:

$$\bigl|O\bigl(E^*, p^0\bigr) \cap U\bigl(G, p^0\bigr)\bigr| > |G|. \tag{10}$$

Because G is an arbitrary non-empty subset of E^*, condition (10) means that the set E^* is in excess demand at prices p^0. Hence, E_0^* cannot be maximal.

Finally, any path $(p^0, \dots, p^n)$ with distance $d_\infty(p^0, p^n) = k$ can be divided into k numbers of paths where the Chebyshev distance equals one in each path. From the above findings, we know that for each of these k paths, there has been exactly one E^*-increase. Consequently, there must be exactly k number of E^*-increases in total. □

For given valuations v, let $I_{VE}(v)$, $I_{VD}(v)$, $I_{E^*S^*}(v)$ and $I_{S^*E^*}(v)$ denote the number of iterations before convergence to the VCG prices for the Vickrey–English auction, the Vickrey–Dutch auction, the E^*S^*-path of VED, and the S^*E^*-path of VED, respectively. We now have the following corollaries to Theorem 4.

Corollary 1. *The number of iterations to convergence at the VCG prices for the Vickrey–English–Dutch auctions is given by* $I_j(v) = d_\infty(p^s, p^n) + d_\infty(p^n, p^{\mathrm{VCG}})$ *for* $j \in \{E^*S^*, S^*E^*\}$.

Proof. By Theorem 4, the first term $d_\infty(p^s, p^n)$ gives the number of E^*-increases (S^*-decreases) until $E_t^* = \emptyset$ ($S_t^* = \emptyset$), and the second term $d_\infty(p^n, p^{\mathrm{VCG}})$ gives the number of S^*-decreases (E^*-increases) until also $S_t^* = \emptyset$ ($E_t^* = \emptyset$). Lemma 1 and Theorem 1 then prove convergence to the VCG prices. □

Corollary 2. *The number of iterations to convergence at the VCG prices for the Vickrey–English auction and the Vickrey–Dutch auction is given by* $I_j(v) = d_\infty(p^0, p^{\mathrm{VCG}})$ *for* $j \in \{VE, VD\}$.

Proof. Because S_t^* (E_t^*) is empty in the entire iterative process for the Vickrey–English auction (Vickrey–Dutch auction), and because the Vickrey–English auction (Vickrey–Dutch auction) consists only of E^*-increases (S^*-decreases), the result follows directly from Theorem 4. □

4.1. The greedy Vickrey–English–Dutch auction

The basic idea in the Vickrey–English–Dutch auction is to eliminate all (non-empty) maximal sets in excess demand first, and then to eliminate all (non-empty) sets in excess supply, or vice versa. This approach is motivated by the observation that if prices are adjusted in both these sets simultaneously, the iterative process may get trapped into a cycle (see Example 5).[10] Of course, there are rules that force the iterative process to leave the cycle and, at the same time, guarantee convergence to the VCG prices. One example of this is the following algorithm.

Algorithm 2. Initialize the price vector to the starting prices $p^s \in \mathbb{N}^{m+1}$. For each iteration $t = 0, 1, \ldots$:

1. Collect the demand sets $D_b(p^t)$ of every bidder $b \in B$.
2. If $E_t^* = S_t^* = \emptyset$ at p^t, terminate the algorithm. If $p^t = p^{t-2}$ for some $t \geqslant 2$, set $p^t = p^s$ and run Algorithm 1. Otherwise, goto Step 3.
3. Identify the sets E_t^* and S_t^*. Let $p_i^{t+1} = p_i^t + 1$ if $i \in E_t^*$, $p_i^{t+1} = p_i^t - 1$ if $i \in S_t^*$, and $p_i^{t+1} = p_i^t$ otherwise. Set $t := t + 1$ and start a new iteration from Step 1.

Algorithm 2 is a greedy version of VED (consequently called Greedy VED, henceforth) where the prices are adjusted for all items in the maximal set in excess demand and all items that are in excess supply until the iterative process gets trapped into a cycle (i.e. when $p^t = p^{t-2}$). If this occurs, VED is adopted. As illustrated in the next section, Greedy VED will, on average, require fewer iterations than VED in the numerical simulations.

5. Simulation results

This section presents computational results obtained by numerical simulations. It is assumed that there are five items (not counting the null-item) and $|B| \in \{5, 6, 7, 8, 9, 10, 15, 20, 25, 30, 40, 50\}$ bidders. As already explained in Section 2, the values v_{bi} are distributed according to some discrete probability density function f. The support of f is given by $[0, 100]$, and the probability of a zero valuation is 25 percent for each item.[11] In the numerical experiments, the values in $[1, 100]$ are distributed according to one of three different distributions. The first is the uniform distribution (UNI). The other two distributions have the same support and expected value as the uniform distribution, and are given by discrete truncated normal distributions with standard deviations 10 and 50 (Norm10 and Norm50, respectively).[12] The starting prices for VED are based on the average VCG prices (rounded to the closest integer) in a simulation with 1000 repetitions for each problem size.[13] For example, when $(|B|, |I|) = (5, 5)$ the simulations generated the mean VCG prices $\bar{p}^{\mathrm{VCG}} = (13, 12, 12, 12, 12)$. The

[10] This need not be the case. In Example 6, for example, only two iterations are required before convergence to the VCG prices if the price for item 1 is decreased by one unit *and* the price for item 2 is increased by one unit in each iteration.

[11] This reflects the situation that not all items are valuable for all bidders. Mishra and Parkes (2009) adopt the same assumption in their simulation study.

[12] We assume that the standard deviations are "large" because if they are "small" (say between 1 and 5), VED outperforms the Vickrey–English auction and the Vickrey–Dutch auction in almost 100 percent of all problems, and requires 85–97 percent fewer iterations on average. To avoid this, standard deviations are assumed to be "large".

[13] It is in general very hard to calculate the expected VCG price for a given distribution as there are $(u+1)^{|B|\times|I|}$ number of possible valuation profiles where u represents the upper limit in the support. Hence, it is convenient to rely on simulations to find an approximation of the expected VCG prices.

Table 1
Fraction of auctions (aggregated over all problem sizes) where the E^*S^*-path of VED is equally fast ($=$) or strictly faster ($<$) than the Vickrey–English auction (VE) and the Vickrey–Dutch auction (VD) in terms of required iterations before convergence.

Distribution	= VE	< VE	= VD	< VD
UNI	0.0104	0.8704	0.0517	0.9263
Norm10	0.0023	0.9113	0.0000	1.0000
Norm50	0.0083	0.8804	0.0276	0.9598
Aggregated mean	0.0070	0.8874	0.0264	0.9620

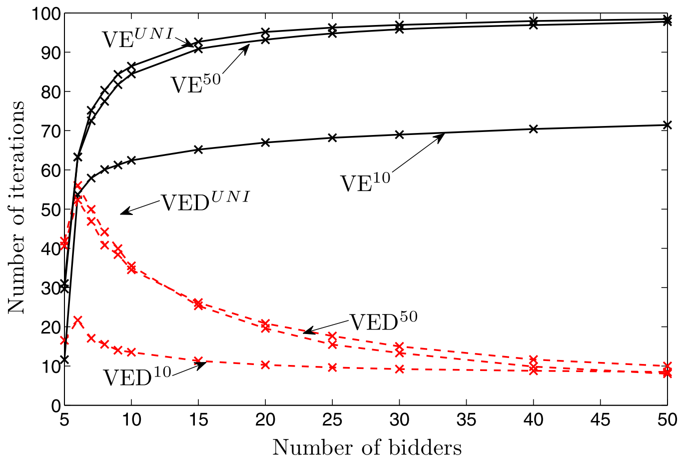

Fig. 2. The mean number of iterations for the E^*S^*-path of VED and the Vickrey–English auction (VE) for the different problem sizes and distributions.

performance of the E^*S^*-path of VED, for starting prices calculated as in the above, will be evaluated by a comparison to the Vickrey–English auction and the Vickrey–Dutch auction.[14]

We start by comparing E^*S^*-path of VED to the Vickrey–English and the Vickrey–Dutch auction. Table 1 reports the fraction of auctions (aggregated over all problem sizes) where the E^*S^*-path of VED is equally fast or strictly faster than the Vickrey–English auction and the Vickrey–Dutch auction in terms of required iterations before convergence. For example, for the uniform distribution, the E^*S^*-path of VED is equally fast (strictly faster) as (than) the Vickrey–English auction in 1.04 percent (87.04 percent) of all cases. The E^*S^*-path of VED is therefore weakly faster than the Vickrey–English auction in $1.04 + 87.04 = 88.08$ percent of all investigated problems for the uniform distribution. In fact, this is the case where VED performs worst in comparison to the Vickrey–English auction and the Vickrey–Dutch auction.

Fig. 2 compares the mean number of iterations for the E^*S^*-path of VED to the Vickrey–English auction. As can be seen from the figure, VED clearly outperforms the Vickrey–English auction except when there are very few bidders. This is natural as few bidders imply low competition, leading to low prices. Hence, the VCG prices are very close to the sellers reservation prices or, equivalently, the starting prices of the Vickrey–English auction. Similarly, when comparing VED to the Vickrey–Dutch auction (Fig. 3), the differences are small when there are many bidders. The same intuition applies in this case, i.e., when there are many bidders, the VCG prices are pushed towards the maximal valuations or, equivalently, the starting prices of the Vickrey–Dutch auction which naturally reduces the number of required iterations before convergence for the Vickrey–Dutch auction.

Figs. 4 and 5 analyze only the auctions where the E^*S^*-path of VED is strictly faster than the Vickrey–English/Vickrey–Dutch auction, and display how many percent fewer iterations, on average, that the E^*S^*-path of VED needs in comparison to the Vickrey–English/Vickrey–Dutch auction. Note that more than 90 percent of the auctions are included in this sample (see Table 1). Aggregated over all problem sizes, the E^*S^*-path of VED requires on average 70 percent fewer iterations than the Vickrey–English auction, and 45 percent fewer iterations than the Vickrey–Dutch auction.

As is clear from the above simulation results, VED clearly outperforms the Vickrey–English and the Vickrey–Dutch auctions in terms of required iterations before convergence. Next, it is evaluated how the E^*S^*-path of VED performs in comparison to Greedy VED. Fig. 6 displays the average difference in iterations between the E^*S^*-path of VED and Greedy VED. As can be seen from the figure, Greedy VED requires, on average, between 0.334 and 15.50 fewer iterations than the E^*S^*-path of VED, depending on the problem size and the distribution. Expressed in percentage, Greedy VED requires, on average, between 1.512 and 36.95 percentage fewer iterations than the E^*S^*-path of VED, and it is weakly faster (equally fast) in 88.01 percentage (8.520 percentage) of the investigated problems. Because Greedy VED outperforms

[14] The results are almost identical for the S^*E^*-path of VED.

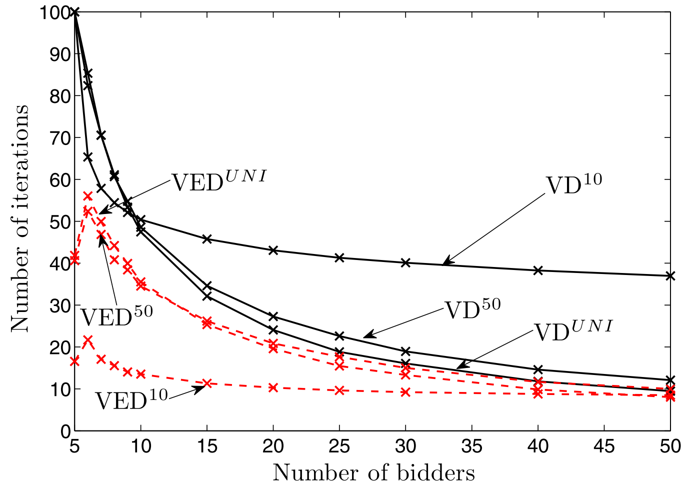

Fig. 3. The mean number of iterations for the E^*S^*-path of VED and the Vickrey–Dutch auction (VD) for the different problem sizes and distributions.

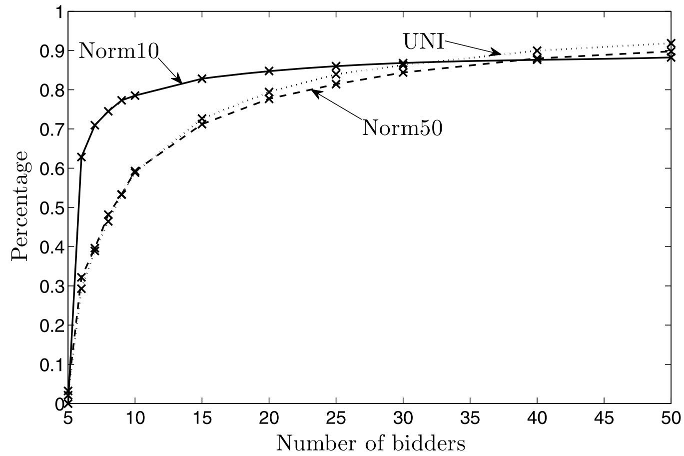

Fig. 4. Percentage fewer iterations required for the E^*S^*-path of VED compared to the Vickrey–English auction given that the E^*S^*-path of VED is faster than the Vickrey–English auction.

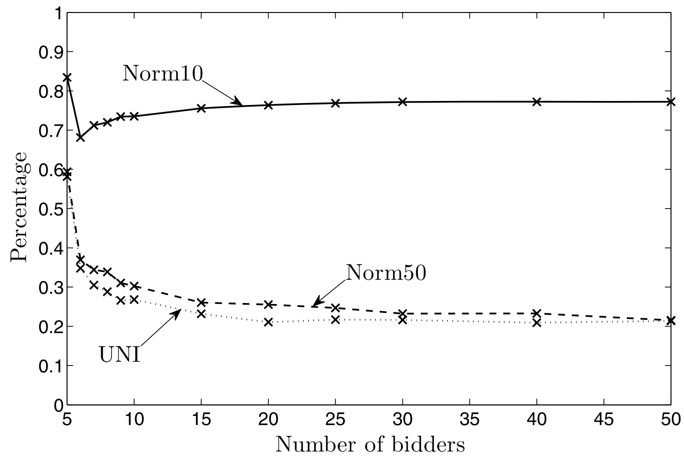

Fig. 5. Percentage fewer iterations required for the E^*S^*-path of VED compared to the Vickrey–Dutch auction given that the E^*S^*-path of VED is faster than the Vickrey–Dutch auction.

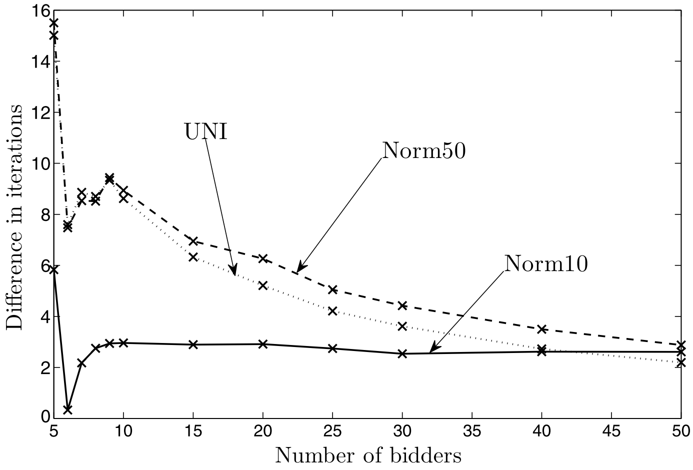

Fig. 6. Average difference in iterations between the E^*S^*-path of VED and Greedy VED for the different problem sizes and distributions.

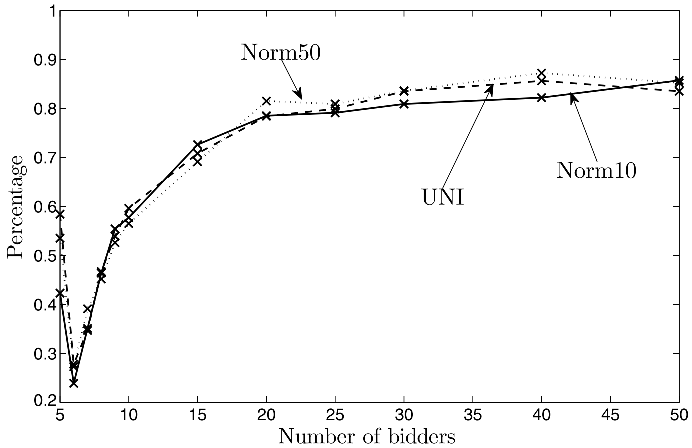

Fig. 7. Percentage of the problems where the required iterations for Greedy VED equal the required iterations for The Shortest Path.

the E^*S^*-path of VED (and, consequently, also the Vickrey–English and the Vickrey–Dutch auctions), it is interesting to evaluate its relation to The (theoretically) Shortest Path between the starting prices p^s and the VCG prices p^{VCG}, i.e., $d_\infty(p^s, p^{\mathrm{VCG}}) = \max_i(|p_i^s - p_i^{\mathrm{VCG}}|)$. The simulation results reveal that the path prescribed by Greedy VED is identical to The Shortest Path in 62.89 percentage of the investigated problems. Fig. 7 offers a more detailed analysis for the different distributions and problem sizes. A general observation from the figure is that the path generated by Greedy VED is remarkably often identical to The Shortest Path considering the assumed huge standard deviations of the value distributions (see footnote 12).

Note, finally, that the numerical experiments presented in this section have only considered distributions with at most one "peak" and where the expected value is in the center of the support. A relevant question is then: what if there are several "peaks" in the distribution and/or if the expected value is non-centered? As far as we can see from our simulations studies (not presented in this paper), VED will perform significantly better, on average, than the Vickrey–English auction and the Vickrey–Dutch auction even if the distributions not are standardized as in this study.

6. Conclusions

The main purpose of the paper was to provide a polynomial time iterative multi-item auction that always locates the VCG prices independently of its starting point. This auction may use information (value distributions, previous auctions, historical bidding behavior, etc.) about the bidders (if available) to significantly reduce the number of required iterations before convergence. This was also illustrated by means of numerical simulations. Even if the purpose of this paper was not to identify the fastest algorithm that is allowed to start at any price vector in the price space and yet locate the VCG prices, the simulation studies demonstrated that the path selected by Greedy VED is remarkably close to The Fastest Path.

Finally, the proposed Vickrey–English–Dutch auction was also demonstrated to be significantly faster than the Vickrey–English and the Vickrey–Dutch auction. Of course, one may argue that the comparison between the Vickrey–English–Dutch auction, on the one hand, and the Vickrey–English auction and the Vickrey–Dutch auction, on the other hand, is unfair because the latter two auction formats cannot use information about the bidders as they must start at the lowest and the highest possible prices in the price space, respectively, independently of the available information. Another way of looking at it is that the findings in these comparisons highlights the importance of future work in this area.

Appendix A. Proof of Theorem 2

To prove Theorem 2, note first that it is clear that there exists a set in positive excess demand $\tilde{S}$ with maximal cardinality as the set is allowed to be empty. Thus we need only prove that it is unique. This proof demonstrates that the set $\hat{S} \subseteq I^+(p)$ of all universally allocated items (Definition 1) with positive prices is identical to the set $\tilde{S}$. The conclusion then follows as $\hat{S}$ is unique (Mishra and Parkes, 2009). In the proof, we also note that, at any given price vector p, the set $X(D(p))$ is non-empty by construction. That is, there always exists some provisional assignment x.

It is first demonstrated that $\tilde{S} \subseteq \hat{S}$. If $\tilde{S} = \emptyset$, then $\tilde{S} \subseteq \hat{S}$ by construction. Suppose therefore that $\tilde{S} \neq \emptyset$. To prove the statement, we need to demonstrate that each item $j \in \tilde{S}$ is universally allocated. Because the set $\tilde{S}$ is in positive excess demand, Definition 5 gives:

$$\left|U(T,p) \cap O^+(\tilde{S},p)\right| > |T| \quad \text{for each non-empty } T \subseteq \tilde{S}. \tag{A.1}$$

Consider now the bidder b_0 with $x_{b_0} \in \tilde{S}$. Hence, we need to demonstrate that $x_{b_0} \in \tilde{S}$ is universally allocated. By condition (A.1), there is a bidder $b_1 \in O^+(\tilde{S},p)$, $b_1 \neq b_0$, with $x_{b_0} \in D^+_{b_1}(p)$. Let $B_0 = B_{-b_0} = B \setminus \{b_0\}$, $y_{b_1} = x_{b_0}$ and $y_j = x_j$ for all $j \in B_0 \setminus \{b_1\}$. Now, if $x_{b_1} \notin A(x)$, then $A_{-b_0}(y) = A(x)$. Thus x_{b_0} is universally allocated. Suppose instead that $x_{b_1} \in A(x)$. But then $x_{b_1} \in \tilde{S}$ as $b_1 \in O^+(\tilde{S},p)$ and x is provisional. Let $T_0 = \tilde{S} \setminus \{x_{b_0}\}$. Consequently, $|(U(\tilde{S},p) \cap O^+(\tilde{S},p)) \cap B_0| > |T_0|$ by construction of the sets B_0 and T_0 together with condition (A.1). Again, there is a bidder $b_2 \in O^+(\tilde{S},p) \cap B_0$ with $x_{b_1} \in D^+_{b_2}(p)$. Let $y_{b_1} = x_{b_0}$, $y_{b_2} = x_{b_1}$ and $y_j = x_j$ for all $j \in B_0 \setminus \{b_1\}$. Now, if $x_{b_2} \notin A(x)$, then $A_{-b_0}(y) = A(x)$ by construction. Thus x_{b_0} is universally allocated. By repeating the above arguments, it is possible to construct an assignment y where $y_{b_1} = x_{b_0}$, $y_{b_2} = x_{b_1}, \ldots, y_{b_k} = x_{b_{k-1}}$, and $y_j = x_j$ for all $j \in B \setminus \{b_1, \ldots, b_k\}$. Note that there must be an index $l \leqslant k \leqslant |\tilde{S}|$ such that $x_{b_l} \notin A(x)$ as condition (A.1) holds for $T = \tilde{S}$. But then $A_{-b_0}(y) = A(x)$ where $y_{b_1} = x_{b_0}$, $y_{b_2} = x_{b_1}, \ldots, y_{b_l} = x_{b_{l-1}}$, and $y_j = x_j$ for all $j \in B \setminus \{b_1, \ldots, b_l\}$ by construction. Hence, x_{b_0} is universally allocated.

It is next established that $\hat{S} \subseteq \tilde{S}$. Let x be a provisional assignment at the price vector p and $W = A(x) \setminus \hat{S}$. Note that $\hat{S} \subseteq A(x)$ by definition. To obtain a contradiction to $\hat{S} \subseteq \tilde{S}$, suppose that $\hat{S} \nsubseteq \tilde{S}$. Then, by Definition 5, there is a non-empty set $T \subseteq \hat{S} \subseteq I^+(p)$ such that $|U(T,p) \cap O^+(\hat{S},p)| \leqslant |T|$. Note first that $|U(T,p)| > T$ as $T \subseteq \hat{S} \subseteq I^+(p)$, i.e., each item $x_j \in T$ must be assigned to some bidder b with $x_j \in D^+_b(p)$ even if only bidders $B_{-j} = B \setminus \{j\}$ are considered since all items in $\hat{S}$ are universally allocated. Consequently, there is a bidder $b \in U(T,p)$ where $b \notin O^+(\hat{S},p)$. We also remark that for this bidder b there is, by definition, an item $i \in I^+(p) \setminus \hat{S}$ such that $i \in D^+_b(p)$ otherwise bidder b would have been included in $U(T,p) \cap O^+(\hat{S},p)$. We next note that the item i must belong to $A(x)$. To see this, suppose that $i \notin A(x)$. If $x_b \notin A(x)$, then $A(y) > A(x)$ for the assignment y where $y_j = x_j$ for all $j \in N \setminus \{b\}$ and $y_b = i$. This contradicts that the maximal number of bidders are assigned an item at the provisional assignment x. If $x_b \in A(x)$, then there exists an assignment y such that $A_{-b}(y) = A(x)$ as item x_b universally allocated. Hence, if $y_b = i$, we again obtain the contradiction $A(y) > A(x)$. Thus, $i \in A(x)$. But then $i \in W$ (i.e. $W \neq \emptyset$) as $i \notin \hat{S}$. But this also means that $x_b \in A(x)$ because if $x_b \notin A(x)$, then $A(x) = A_{-j}(y)$ if $x_j = i$ contradicting the assumption that i not is universally allocated. In summary:

$$\text{if } b \in U(T,p) \text{ but } b \notin O^+(\hat{S},p), \quad \text{then } x_b \in A(x). \tag{A.2}$$

We next demonstrate that if $x_b \in \hat{S}$, then $D^+_b(p) \subset \hat{S}$. To see this, suppose that $j \in D^+_b(p)$ for some $j \notin \hat{S}$. As item x_b is universally allocated there exists an assignment y such that $A_{-b}(y) = A(x)$. Now, if $j \notin A(x)$ then more items are allocated at assignment y than under assignment x if $y_b = j$ which contradicts that x satisfies the maximum number of bidders. If $j \in A(x)$ then $A_{-k}(y) = A(x)$ for $x_k = j$ as x_b is universally allocated which contradicts that $j \notin \hat{S}$. Hence:

$$\text{if } x_b \in \hat{S}, \quad \text{then } D^+_b(p) \subset \hat{S}. \tag{A.3}$$

Consider now the set of bidders $B_{-j} = B \setminus \{j\}$ for some j with $x_j \in T$. Because $D^+_b(p) \subset \hat{S}$ if $x_b \in \hat{S}$, by condition (A.3), we can, without loss of generality, assume $y_b = x_b$ for all $b \in B_j$ with $x_b \in \hat{S}$. Because all items in $T \subset \hat{S}$ are universally allocated and $|U(T,p) \cap O^+(\hat{S},p)| \leqslant |T|$, there must be a bidder $b \in U(T,p)$ where $b \notin O^+(\hat{S},p)$ and $x_b \notin T$ but $y_b \in T$. By condition (A.2), $x_b \in A(x)$. From the assumption $y_b = x_b$ for all $b \in B_j$ with $x_b \in \hat{S}$, it then follows that $x_b \in W$. Now, as x_b not is universally allocated this means that $x_b \notin A_{-j}(y)$. Hence, $A_{-j}(y) < A(x)$, which is a contradiction.

In summary, $\tilde{S} \subseteq \hat{S}$ and $\hat{S} \subseteq \tilde{S}$ implies $\hat{S} = \tilde{S}$, as desired.

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