Comput Econ
DOI 10.1007/s10614-013-9380-0

The Duo-Item Bisection Auction

Albin Erlanson

Accepted: 2 May 2013

Abstract This paper proposes an iterative sealed-bid auction for selling multiple heterogeneous items to bidders interested in buying at most one item. It generalizes the single item *bisection auction* (Grigorieva et al. Econ Theory, 30:107–118, 2007) to the environment with multiple heterogeneous items. We focus on the case with two items for sale. We show that the auction elicits a minimal amount of information on preferences required to find the Vickrey–Clark–Groves outcome (Clarke, Public Choice, XI:17–33, 1971; Groves, Econometrica, 61:617–631, 1973; Vickrey, J Finance, 16:8–37, 1961), when there are two items for sale and an arbitrary number of bidders.

Keywords Bisection auction · Multi-item · Unit-demand · Sealed-bid

JEL Classifiation D44 · C72

1 Introduction

Recent research in auction theory has produced a number of papers on iterative auctions (Mishra and Parkes 2009; Perry and Reny 2005; Ausubel 2004). In an iterative auction the auctioneer announces a price and bidders submit their bids. A new price based upon the reported bids is announced by the auctioneer. The process is repeated until an allocation is determined. This iterative procedure contrasts with the approach taken in direct mechanisms, where bidders submit their preferences and an allocation is determined. There are several reasons for focusing on iterative auctions. In iterative auctions bidders may not reveal all information regarding their private valuations. This could be a beneficial property. It has been shown that full revelation of preferences

A. Erlanson (✉)
Department of Economics, Lund University, Box 7082, 220 07 Lund, Sweden
e-mail: albin.erlanson@nek.lu.se

Published online: 15 May 2013

can be problematic (Rothkopf et al. 1990; Engelberecht-Wiggans and Kahn 1991). Partial revelation of preferences can lead to less communication and thereby decrease the amount of data required for the computation of an allocation. Another argument for looking closer at iterative auctions is the prevalence of them in real world auctions (e.g. English auction, Dutch auction, etc.).

One property of importance when analyzing an auction is efficiency. An efficient assignment maximizes the overall value derived by the bidders. This translates to designing an auction with efficiency as part of the equilibrium in the game induced by the auction. The benchmark for the environment with private valuations is the Vickrey–Clarke–Groves mechanism (Clarke 1971; Groves 1973; Vickrey 1961), henceforth VCG. The VCG mechanism is a direct mechanism with truth-telling as a weakly dominant strategy and the equilibrium outcome is efficient. Another classical mechanism is the English auction. It is an iterative open bid ascending auction for selling one item, and its descending counterpart is the Dutch auction. The English auction is strategically equivalent to the Vickrey auction. Hence, in the English auction it is a weakly dominant strategy to bid truthfully and the resulting equilibrium is efficient.

The single item *bisection auction* presented and analyzed by Grigorieva et al. (2007) is an example of a sealed-bid iterative auction. It elicits a limited amount of information on preferences, but still reaches the VCG outcome. It has also fewer rounds than the English auction. In other words, the single item bisection auction has the correct incentive structure, it is privacy preserving and has a fast convergence rate. For multiple heterogeneous items things get more complicated. It is a complex problem to solve in the most general setting where bidders are allowed to bid on any packages of items. In a seminal paper by Demange et al. (1986) an iterative auction for multiple items with unit-demand bidders was presented. The auction proposed in Demange et al. (1986) results in the VCG outcome.

In this paper we propose a multi-item bisection auction. It generalizes the single item bisection auction (Grigorieva et al. 2007). We keep the assumption from the single item bisection auction of unit-demand bidders. Thus, we do not consider the case when bidders bid on packages of items. The auction to be proposed is a multi-item sealed-bid auction for selling heterogeneous items to bidders interested in acquiring at most one item. They all have private valuations. In other words it is a standard assignment problem. To illustrate the idea behind the multi-item bisection auction it is enough to consider the case with two items for sale. We call this the duo-item bisection auction. The duo-item bisection auction is later modified to allow for a more selective elicitation of information on preferences.

All our results are given for the environment with two items for sale. However, many of them are straightforward to generalize to a setting with more than two items for sale. The first result establishes an upper limit on the number of iterations for the duo-item bisection auction. Thereafter we proceed by showing that the duo-item bisection auction reaches the VCG outcome under the assumption of truthful bidding. This is not as restrictive as it first may look. There are general results on incentives for dynamic auction mechanisms implementing the VCG outcome. Loosely speaking the results establish truthful bidding as a weakly dominant strategy when bidding strategies are constrained to maximize payoff in each step by taking prices as given (Gul and Staccetti 2000; Parkes 2001). Our last result concerns the modification of the duo-item bisection auction. We prove that the modified duo-item bisection auction attains the VCG outcome, whilst eliciting the minimal amount of information on preferences.

The single item bisection auction is straightforward to describe. However, already with two items the generalized bisection auction gets involved and more effort is required to describe it. This is not specific for this auction mechanism. Using auctions for solving an assignment problem with an arbitrary number of items and bidders is a complex problem. There are both computational and theoretical obstacles to overcome. The multi-item bisection auction can also be seen as a computational alternative for solving an assignment problem with known valuations.

The rest of the paper is organized as follows. Sect. 2 begins by describing the single item bisection auction. After this the model is presented together with some preliminaries. In the subsequent third section the generalized multi-item bisection auction is described. This is done by considering the case of two items for sale and discuss the auction mechanism for this scenario. Section 3 ends with a short discussion of the multi-item bisection auction. Section 4 contains the main results and Sect. 5 concludes the paper.

2 Model, Preliminaries and Background

2.1 Single Item Bisection Auction

The single item bisection auction (Grigorieva et al. 2007) can be used for selling one indivisible item to a group of bidders with integer valued valuations drawn from a bounded interval $[0, 2^R)$, for some positive integer R. The bisection auction has R rounds and the starting price in the auction is 2^{R-1}. A bidder in the auction has two options. She can report a *yes* answer stating that she is willing to buy the item at the current price, or report a *no* answer and thereby stating that she is not willing to buy the item at the current price. The price changes through the auction as a function of the reported answers. In case of two or more yes answers the price increases to the middle of the upper interval $[2^{R-1}, 2^R)$. Only bidders with a yes answer in the previous round are allowed to participate in the next round. In the other scenario, where at most one bidder submits a yes answer, the price decreases to the middle of the lower interval $[0, 2^{R-1})$. The active bidders in the next round are those with a no answer. If exactly one bidder reported a yes, she is the winner of the auction and the item is given to her. However, the auction does not terminate. It continues and finds the highest price for which at least one bidder submits a yes answer.

Repeating this process will eventually yield a winner and a price. The price is the integer contained in the half open interval generated at the Rth round in this process. This interval contains exactly one integer, since the interval is decreased by half of its length in each of the R rounds and the initial length of the interval is 2^R. If no round had exactly one yes answer, then the winner is chosen randomly among the active bidders at round R. The only information revealed to the bidders' are the announced prices in each round. No other information is available (Table 1).

The purpose of this paper is to generalize the single item bisection auction to a setting with multiple heterogeneous items. For doing this it is convenient to have a description

Table 1 The single item bisection auction

Round	Price	Bidder α	Bidder β	Bidder γ	Bidder δ
1	8	Yes	Yes	Yes	No
2	12	Yes	No	No	–
3	10	–	No	Yes	–
4	11	–	–	Yes	–

of the single item bisection auction in terms of an algorithm. The generalized bisection auction is going to be based upon the algorithm for the single item case. Let p_L^t and p_H^t denote the lower bound respectively the upper bound in the price interval at iteration t.

Algorithm 1 The single item bisection auction.

Start the process with $p_L^0 = 0$ and $p_H^0 = 2^R$ and $A^0 = N$. For each iteration $t = 1, 2, \ldots, R$:

1. Update $p^t = (p_L^{t-1} + p_H^{t-1})/2$, if $t < R$ collect answers from the set of active bidders in A^{t-1}, else if $t = R$ the auction terminates.
   (i) If two or more bidders report yes, set $p_L^t = p^t$, $p_H^t = p_H^{t-1}$, $A^t = \{i \in A^{t-1} : i \text{ reported yes}\}$, $t = t + 1$ and repeat from step 1.
   (ii) If every bidder report no, set $p_L^t = p_L^{t-1}$, $p_H^t = p^t$, $A^t = A^{t-1}$, $t = t + 1$ and repeat from step 1.
   (iii) If only bidder i report yes, set $p_L^t = p_L^{t-1}$, $p_H^t = p^t$, $A^t = A^{t-1} \setminus \{i\}$, $t = t + 1$ and move on to step 2.
2. Update $p^t = (p_L^{t-1} + p_H^{t-1})/2$, if $t < R$ collect answers from the set of active bidders in A^{t-1}, else if $t = R$ the auction terminates.
   (i) If one or more bidders report yes, set $p_L^t = p^t$, $p_H^t = p_H^{t-1}$, $A^t = \{i \in A^{t-1} : i \text{ reported yes}\}$, $t = t + 1$ and repeat from step 2.
   (ii) If every bidder report no, set $p_L^t = p_L^{t-1}$, $p_H^t = p^t$, $A^t = A^{t-1}$, $t = t + 1$ and repeat from step 2.

Below is in an example illustrating the single item bisection auction using the formulation in Algorithm 1.

Example 1 Suppose there are four bidders α, β, γ, and δ participating in the auction. They have the following valuations for the item: $v_\alpha = 13$, $v_\beta = 9$, $v_\gamma = 11$ and $v_\delta = 6$, drawn from the interval $[0, 16)$. There will be 4 rounds in the auction before the winner and the price can be determined. In all of our examples bidders bid truthfully. A bidder reports yes if her valuation is above or equal to the current stated price, and if the price is above her valuation she reports no.

The process starts with the lower bound of $p_L^0 = 0$ and the upper bound of $p_H^0 = 16$. Every bidder belong to the set of active bidders in the first round, i.e. $A^0 = \{\alpha, \beta, \gamma, \delta\}$. In the first round the price is set to $p^1 = (p_L^0 + p_H^0)/2 = 8$ and answers are collected from every bidder. There are three yes answers and one no answer in the first round. With more than two yes answers we end up at step 1.(i). First the price bounds are updated, $p_L^1 = p^0 = 8$ and $p_H^1 = p_H^0 = 16$, then the set of active bidders for the next round is formed. It consists of those with a yes answer, using the notation from Algorithm 1, $A^1 = \{\alpha, \beta, \gamma\}$. The second round starts at step 1 with updating the price $p^2 = (p_L^1 + p_H^1)/2 = 12$ and then collecting answers from bidders in A^1. At the price of 12, α is the sole bidder with a yes answer, and α is the winner of the auction. We are now at step 1. (iii), the price bounds are updated $p_L^2 = p_L^1 = 8$ and $p_H^2 = p^1 = 12$. Round number 3 begins at step 2 in Algorithm 1, the price is updated to $p^3 = (p_L^2 + p_H^2)/2 = 10$ and answers are collected from β and γ. With one yes answer from γ and one no answer from bidder β, we are in 2. (i) and the price bounds are updated, $p_L^3 = p^3 = 10$, $p_H^3 = p_H^2 = 12$. In the fourth and last step the price is updated to $p^4 = (p_L^3 + p_H^3)/2 = 11$. A yes answer is collected from γ, the only bidder still active, and the auction terminates. bidder α is the winner of the auction. She gets the item and pays the price of 11. This is the VCG outcome.

2.2 Model and Preliminaries

Let the finite sets of *items* and *bidders* be denoted by $M = \{1, \ldots m\}$ and $N = \{1, \ldots n\}$, respectively. Denote by v_{ij} bidder i's *valuation* of item j. Valuations are assumed to be non-negative integers drawn from a distribution function with support on $(0, 2^R]$, for some positive integer R. Every bidder $i \in N$ knows her own valuation for each item in M. Valuations are i.i.d. random variables. Construct the matrix V of all valuations v_{ij} with n rows and m columns. Row i contains bidder i's valuations and column j contains all valuations for the jth item. The payoff for bidder i for obtaining item $j \in M$ at the price p_j is given by $v_{ij} - p_j$.

There is a null-item denoted by 0. The value and the price of it is zero and it can be assigned to any number of bidders. For notational convenience let $\tilde{M} = M \cup \{0\}$. The standard notation of N_{-i} is employed, it stands for the set N excluding bidder i. The demand correspondence for bidder i at price vector p is defined as,

$$D_i(p) = \{j \in \tilde{M} : v_{ij} - p_j \geq v_{ij'} - p_{j'} \text{ for all } j' \in \tilde{M}\}.$$

The model with multiple heterogeneous indivisible items and bidders consuming at most one item is a standard *assignment* problem. Applications of the model includes for example housing markets. Using auctions is one way to solve the assignment problem and obtain an allocation. Let the pair $(x, p) \in \tilde{M}^n \times \mathbb{N}_+^{m+1}$ denote an allocation. The first component x is an assignment and p is a price vector, where x_i indicates which object bidder i is assigned and p_j is the price of object j. An allocation is *feasible* if:

(i) $x_i \neq x_{i'}$ for all $i \neq i'$ and $x_i, x_{i'} \in M$
(ii) if $p_j > 0$ then $x_i = j$ for some $i \in N$.
An allocation (x, p) is *efficient* if:
$\sum_{i \in N} v_{ix_i} \geq \sum_{i \in N} v_{iy_i}$ for all feasible allocations (y, p).

There are several possible ways to measure the degree of preference elicitation. In this paper a sort of binary measure is employed. This approach resembles the one taken in Andersson and Andersson (2012) and Hudson and Sandholm (2004). They also measure the degree of elicitation of preferences in relation to full revelation. A bidder's valuation for an item is considered elicited if the exact value is known to the auctioneer. The total number of valuations are nm, full revelation of preferences means that all nm valuations are elicited, and the measure equals one. At the other extreme with no information at all about preferences the value of the measure is zero. In all other cases the measure lies between zero and one.

Definition 1 The measure on preference elicitation is defined as the ratio between the number of elicited valuations and the total number of valuations nm.

The multi-item bisection auction to be proposed elicits valuations from one item at a time, and then an allocation is computed after the last round. The auction mechanism can be viewed as a method of iteratively eliciting information on preferences to compute the VCG outcome. Let us formalize what we mean by iteratively eliciting information on preferences.

Definition 2 Let E be an $n \times m$ matrix consisting of only zero entries. For each $j \in M$ construct a sequence of vectors $(x^{j,k})_{k=0}^{m}$, where each vector $x^{j,k} \in \mathbb{R}^n$ and $x^{j,0}$ is equal to the zero vector of dimension n. For each elicited valuation k , $1 \leq k \leq m$, let $x_i^{j,k} = v_{ij}$ and for each $h \neq i$, $x_h^{j,k} = x_h^{j,k-1}$. When the elicitation of valuations for item j is terminated, at step $k \leq m$, let the jth column of E be given by $x^{j,k}$. This formalizes what we call a method of iteratively eliciting information on preferences.

3 A Generalized Bisection Auction

3.1 Duo-Item Bisection Auction

The duo-item bisection auction is a sealed-bid auction for selling two items. It is presented in terms of an algorithm. The algorithm generates price vectors and an allocation is computed based upon the bidders' answers in the process. Call the two items for sale 1 and 2. Further, denote by p_1 and p_2 the final prices to be generated in the auction. The idea of obtaining prices by decreasing the length of intervals is maintained. As stated before we let p_{1L}^t denote the lower bound of the price for item 1 at iteration t and p_{2L}^t denotes the lower bound for item 2 at iteration t. The corresponding upper bounds for item and 1 and 2 at iteration t are denoted p_{1H}^t and p_{2H}^t.

In order to be able to reach the VCG-outcome it is necessary to elicit more information on preferences than in the single item case. To elicit more information the single item bisection auction will be divided into different sub-processes. Each sub-process is a continuation of the original process and a unique price is generated after in total R iterations, counting from the beginning of the auction. The division into new sub-processes can occur several times and it always creates two new processes. We will call this division of the process a *split* and it is defined as follows.

Definition 3 A *split* in Algorithm 1 at step t divides the process in two parts. Process 1 consists of bidders with a yes answer for item j at price p_j^t, the updated price bounds are $p_{jL}^t = p_j^t$ and $p_{jH}^t = p_{jH}^{t-1}$. The active bidders at step t with a no answer for item j are in process 2, the price bounds are $p_{jL}^t = p_{jL}^{t-1}$ and $p_{jH}^t = p_j^t$. Each process continues from iteration $t+1$.

The bidders in the process with a yes answer at the split are called yes-bidders and similarly bidders with a no answer are called no-bidders. The auctioneer does not inform the bidders about the split, and they cannot infer that a split has occurred from only observing the price path in the auction. This is possible since it is a sealed-bid auction and the only information they receive is the announced price. In both processes prices are updated based on the rules of the single item bisection auction, and it is not possible to differentiate among the two processes.

For the single item bisection auction it was sufficient to find one price. Here we need two prices, one for item 1 and another for item 2. Building on the process in Algorithm 1 the duo-item bisection auction is constructed. The idea is to use Algorithm 1 for finding prices for item 1 and 2 separately. And by combining Algorithm 1 with splits several prices for each item can be generated, and a sufficient amount of information about bidders' preferences can be elicited.

Algorithm 2 The duo-item bisection auction.

Start the auction with $p_L^0 = (p_{1L}^0, p_{2L}^0) = (0, 0)$, $p_H^0 = (p_{1H}^0, p_{2H}^0) = (2^R, 2^R)$, $A^0 = N$, and initialize Algorithm 1 with item 1 and keep the price for item 2 fixed at 2^R.

1. Check for each iteration in Algorithm 1 after step 1:
   (i) If one bidder answered yes, split the auction and go to 2.
   (ii) If two bidders answered yes, split the auction and go to 3.
   (iii) Otherwise keep on iterating Algorithm 1.
2. Process 1 with the yes-bidder continue to step 2 in Algorithm 1, call the price generated $p_1^R(1)$. Process 2 remains at step 1 in Algorithm 1. If one bidder reports yes another split is made, else keep on iterating and call the price generated $p_1^R(2)$. If a split occurs both processes continue to step 2 in Algorithm 1. Let $p_1^R(2)$ be the price from the first process and $p_1^R(3)$ from the other.
3. Process 2 with the no-bidders continue to step 2 in Algorithm 1, call the price generated $p_1^R(3)$. Process 1 remains at step 1 in Algorithm 1. If one bidder reports yes another split is made, else keep on iterating call the price generated $p_1^R(1)$. If a split occurs both processes continue to step 2 in Algorithm 1. Let $p_1^R(1)$ be the price from the first process and $p_1^R(2)$ from the other.
4. Start Algorithm 1 with item 2 and set the price for item 1 to 2^R. Repeat step 1 to 3 from above.

Now, using the information from Algorithm 2 an allocation consisting of a price vector and an assignment can be determined. It will be shown later in the paper that the process in Algorithm 2 can be viewed as a method of successively eliciting the information necessary to compute the VCG outcome.

Algorithm 2 generates at most three different prices for each item and three associated sets of bidders. This follows from the fact that each process in Algorithm 2 leads to a distinct price $p_j^R(k)$, with $j = 1, 2$ and $k = 1, 2, 3$. Denote the set of winners in each separate process by W_j^k. The set of winners for each process consist of bidders with a yes answer for the highest associated price in their specific process. For example if the first split is with one bidder, then the set of winners equals this bidder with the yes answer. Further, let $W_j = W_j^1 \cup W_j^2 \cup W_j^3$ for $j = 1, 2$, and define a function $f : W_1 \times W_2 \longmapsto \mathbb{N}$ as $f(i, j) = p_1^R(i) + p_2^R(j)$. Now the allocation can be defined.

The assignment is determined as follows,

Table 2 Initializing the duo-item bisection auction and the first split for item 1

Round	Price	Bidder α	Bidder β	Bidder γ	Bidder δ
1	8	Yes	Yes	Yes	No
2	12	Yes	No	No	–
3	14	No	–	–	–
4	13	Yes	–	–	–

Table 3 The second split for item 1 in the duo-item bisection auction

Round	Price	Bidder β	Bidder γ
3	10	No	Yes

$$x_{i'} = 1,\ x_{j'} = 2, \text{ where } (i', j') \in \underset{(i,j)\in W_1\times W_2}{\arg\max} f(i, j) \text{ and } x_k = 0 \text{for all other} k \in N. \tag{1}$$

The prices of item 1 and 2 are,

$$\begin{aligned} p_1 &= \max_{(i,j)} f(i, j) - p_2^R(j'),\ \text{ where } (i, j) \in (W_1 \setminus \{i'\}) \times (W_2 \setminus \{i'\}) \\ p_2 &= \max_{(i,j)} f(i, j) - p_1^R(i'),\ \text{ where } (i, j) \in (W_1 \setminus \{j'\}) \times (W_2 \setminus \{j'\}). \end{aligned} \tag{2}$$

Below is an example of how the duo-item bisection auction works.

Example 2 Consider a situation with the same bidders as in Example 1, and add a second item. Call the item from the first example item 1 and the new item 2. The bidders α, β, γ and δ have private valuations for item 1 and 2. Collect the valuations for item 1 and 2, in a matrix V.

$$V = \begin{pmatrix} v_{\alpha 1} & v_{\alpha 2} \\ v_{\beta 1} & v_{\beta 2} \\ v_{\gamma 1} & v_{\gamma 2} \\ v_{\delta 1} & v_{\delta 2} \end{pmatrix} = \begin{pmatrix} 13 & 4 \\ 9 & 9 \\ 11 & 7 \\ 6 & 5 \end{pmatrix}$$

The two first price updates in the duo-item bisection auction are identical to the single item case. In the test after the second price update at step 1 in Algorithm 2 the condition in 1.(i) is satisfied and the first split is made. After the split we move on to step 2 in Algorithm 2. Process 1 with the yes-bidder α continues to step 2 in Algorithm 1 with the price bounds updated to $p_L^2 = p^2 = 8$, $p_H^2 = p_H^1 = 16$. Henceforth process 1 follows the rules of Algorithm 1 and a price of 13 is found in round 4, the path is displayed in Table 2. The price $p_1^4(1) = 13$ found equals bidder α's valuation for item 1.

Process 2 with the no-bidders β and γ stays at step 1 in Algorithm 1 and the price bounds are updated to $p_L^2 = p_L^1 = 8$, $p_H^2 = p^2 = 12$. The price in round 3 is updated to 10 and this leads to a new split, see Table 3 below.

Table 4 Last rounds for item 1 in the duo-item bisection auction

Round	Price	Bidder γ
4	11	Yes
4	9	Yes

Table 5 Initializing the duo-item bisection auction for item 2 and the first split

Round	Price	Bidder α	Bidder β	Bidder γ	Bidder δ
1	8	No	Yes	No	No
2	12	–	No	–	–
3	10	–	No	–	–
4	9	–	Yes	–	–

Table 6 The second split for item 2 in the duo-item bisection auction

Round	Price	Bidder α	Bidder γ	Bidder δ
2	4	Yes	Yes	Yes
3	6	No	Yes	No

Table 7 Last rounds in the duo-item bisection auction for item 2

Round	Price	Bidder α	Bidder δ	Round	Price	Bidder γ
4	5	No	Yes	4	7	Yes

In the process with the yes-bidder γ the price bounds are updated to $p_L^3 = p^3 = 10$, $p_H^3 = p_H^2 = 12$ and the price announced is 11. With the price of 11 γ reports yes and we have found her valuation for item 1. The other process with the no-bidder β ends up in a price of 9, which equals her valuation for item 1. Table 4 below shows the final step with bidder γ and β leading to the two separate prices.

At this point in the duo-item bisection auction we have obtained the three highest valuations for item 1 and can continue with item 2. The procedure in Algorithm 2 is the same for item 2 therefore the various steps are not described as carefully. Table 5 summarizes the path for finding the highest valuation for item 2, it equals the price $p_2^4(1) = 9$.

The first split was made already in iteration 1 after the first price announcement for item 2. All bidders except β reported no when the price was announced to 8 and they moved on to a new separate process. Table 6 below shows the two first rounds in this process. In the second round of this new process the price is announced to 6 and yet another split occurs.

After the final split the price is set to 5 in the process with the no-bidders α and δ, and it is set to 7 with the yes-bidder γ. Now the three highest valuations for item 1 and 2 are elicited and an allocation can be determined as described by Eqs. (1) and (2).

In the resulting allocation item 1 is given to α and she pays the price of 11, and item 2 is given to β for a price of 7. This corresponds to the VCG outcome.

3.2 Modified Duo-Item Bisection Auction

In this section a modification of the duo-item bisection auction is presented. This modified version will be shown to extract the minimal amount of information on preferences required to compute the VCG outcome. The procedure in this auction is same as before. Algorithm 3 will generate prices and sets of potential winners for items 1 and 2. Using these an allocation is determined in the same way as for the standard duo-item bisection auction defined by Eqs. (1) and (2). The only difference between Algorithms 2 and 3 is that the latter elicits less information on preferences without giving up the possibility for the VCG outcome (Table 7).

Before defining the modified duo-item bisection auction let us recall the meaning of sets denoted by W_j^k. They were introduced in Sect. 3.1. Take for example the set W_j^1. It consists of bidders with a yes answer for the highest associated price in the elicitation process for item j.

Algorithm 3 The modified duo-item bisection auction.

1. Run Algorithm 2 until item 2 has passed step 1. Generate W_2^1 and the second highest price for item 2.
2. If $|W_1^1 \cup W_2^1| = 1$ continue in Algorithm 2 and generate $p_2^R(1)$. Further if $p_1^R(1) + p_2^R(2) \geq p_1^R(2) + p_2^R(1)$ or $|W_1^2 \cup W_2^2| = 1$, continue in Algorithm 2 and generate the third highest price for item 2.
3. If $|W_1^1 \cup W_2^1| = 2$ consider the two cases.
   (i) If $|W_2^1| = 2$ continue in Algorithm 2 and elicit $p_2^R(2)$.
   (ii) If $|W_2^1| = 1$ there are two cases depending on $|W_1^1|$.
      a) For $|W_1^1| = 1$, generate $p_2^R(1)$ if $|W_1^2 \cup W_2^1| = 1$ and generate $p_2^R(3)$ if $|W_1^1 \cup W_2^2| = 1$.
      b) For $|W_1^1| = 2$, generate $p_2^R(1)$ and generate $p_2^R(3)$ if $|W_1^1 \cup W_2^2| = 2$.
4. If $|W_1^1 \cup W_2^1| \geq 3$ the auction terminates.

Example 3 This example illustrates the modified duo-item bisection auction. The setup is the same as in example 2, we continue with the four bidders and their valuations. Nothing changes for item 1 in the modified duo-item bisection auction, the change is for item 2. In the first step of Algorithm 3 the price for item 2 is set to 8 and answers are collected. The only bidder with a yes answer is β, and thereby W_2^1 is generated. Now it remains to find the second highest price $p_2^4(2)$. The auction splits and every bidder with a no answer in the first round moves to a new process, which is illustrated in Table 8 below.

Now, Step 1 in Algorithm 3 is completed with the second highest price generated, and we continue in Algorithm 3. There are two different bidders with the highest

Table 8 The modified duo-item bisection auction

Round	Price	Bidder α	Bidder γ	Bidder δ
2	4	Yes	Yes	Yes
3	6	No	Yes	No
4	7	–	Yes	–

valuation for item 1 and 2. This makes step 3 the appropriate next step in Algorithm 3. There are three different scenarios at step 3. With $|W_1^1| = |W_2^1| = 1$ we end up in the middle (ii), since both inequalities are violated there is no need to generate more prices and Algorithm 3 ends here. The final part in the auction is identical to example 2, hence bidder α is given item 1 to the price of 11 and β buys item 2 for the price of 7. The modified duo-item bisection auction neither elicited the highest nor the third highest price for item 2. In terms of iterations the modified version saved 4 iterations out of 15 in total.

3.3 Multi-Item Bisection Auction

Extending the duo-item bisection auction to a situation with more than 2 items for sale can be done in the following manner. Suppose there are $m > 2$ items to allocate in the auction. Instead of eliciting the 3 highest valuations for item 1 and 2, the multi-item bisection auction elicits the $m+1$ highest valuations for all m items, assuming $n > m$. The method is the same as before and it builds on splitting up the auction into separate processes. Similarly to the environment with 2 items for sale the auction begins with item 1 and generates the $m + 1$ highest valuations for item 1. The first split is made when there is a yes answer from $j < m+1$ bidders. The jth highest valuations for item 1 are elicited in the group with yes-bidders and new splits are made if necessary. The remaining valuations to get the $m+1$ highest valuations for item 1 are elicited from the group of no-bidders from the first split. This procedure continues item per item until the $m+1$ highest valuations for item m are elicited. The allocation is determined in an analogous manner to the duo-item bisection auction. Based upon the prices' generated from the auction an efficient assignment is found. The price for item j is determined by calculating the difference between the efficient assignment in N and N_{-i}, where bidder i is the bidder assigned item j.

The main advantages with the multi-item bisection auction vanish when the number of bidders and the number of items are approximately of the same size. Then the multi-item bisection auction basically reveals all information on preferences, and it would have been better to use a direct mechanism from the beginning. The remedy of this would be to create a modification similar to the environment with two items for sale. However, creating such a modification is not as easily done for the general case with m items. The number of possible combinations for efficient assignments grows exponentially with the number of items and it is difficult to find clear cut conditions as with the duo-item auction. The main contribution of the multi-item bisection auction in this context is as an alternative way of solving the standard assignment problem.

4 Main Results

Before presenting the main results a family of partitions of the bidders needs to be introduced. These partitions makes it possible to keep track of the various cases that will be of importance for the following proofs. Informally speaking these partitions sorts bidders according to their valuations of the various items. Based on this sorting the VCG outcome can be assured.

The bidders in the set N can be partitioned into m partitions, one partition T_j for each item $j \in M$. Each partition T_j is based on valuations among the bidders for item j. It is constructed in the following manner. Define for each item $j \in M$ a set $T_{j,1}$, consisting of all bidders (could be one or more) with the highest valuation of item j, i.e. $T_{j,1} = \arg\max_{i \in N} v_{ij}$. This gives the first subset $T_{j,1}$ in the partition T_j. Consecutively define the subset $T_{j,2}$ in the partition T_j consisting of bidders (could be one or more) with the highest valuation for item j amongst remaining bidders in $N \setminus T_{j,1}$, formally $T_{j,2} = \arg\max_{i \in N \setminus T_{j,1}} v_{ij}$. Repeating this procedure until all bidders are assigned into a subset $T_{j,k}$ creates the partition T_j. For each step in the process the group of bidders not yet assigned into a subset is decreasing with at least one. Subset $T_{j,k}$ in the partition T_j consists of bidders with the highest valuation for item j among the remaining bidders not yet assigned into a subset at step $1, 2, \ldots, k-1$. To express the subsets formally in partition T_j, let $T_j^{k-1} = \cup_{m=1}^{k-1} T_{j,m}$. Now, the subset $T_{j,k}$ can be written as $T_{j,k} = \arg\max_{i \in N \setminus T_j^{k-1}} v_{ij}$. Denote by i_{jk} a bidder belonging to $T_{j,k}$. Thus, bidder i_{jk} has the $|T_j^{k-1}| + 1$ highest valuation for item j.

In an auction with two items for sale and at least two bidders there are three possible efficient allocations. In case of several efficient allocations a tie-break rule that randomly picks one of them is used. These are the three scenarios, and they will be used extensively in the following proofs.

(I) $|T_{1,1} \cup T_{2,1}| \geq 2$ with x given by, $x_{i_{11}} = 1$, $x_{i_{21}} = 2$ and $x_i = 0$ for all $i \in N \setminus \{i_{i_{11}}, i_{i_{21}}\}$

(II) $|T_{1,1} \cup T_{2,1}| = 1$ and $v_{i_{11}1} + v_{i_{22}2} \geq v_{i_{12}1} + v_{i_{21}2}$ with x given by, $x_{i_{11}} = 1$, $x_{i_{22}} = 2$ and $x_i = 0$ for all $i \in N \setminus \{i_{i_{11}}, i_{i_{22}}\}$

(III) $|T_{1,1} \cup T_{2,1}| = 1$ and $v_{i_{11}1} + v_{i_{22}2} < v_{i_{12}1} + v_{i_{21}2}$ with x given by, $x_{i_{12}} = 1$, $x_{i_{21}} = 2$ and $x_i = 0$ for all $i \in N \setminus \{i_{i_{12}}, i_{i_{21}}\}$

As pointed out earlier the single item bisection auction has R iterations. The number of iterations required in the duo-item bisection auction varies, but there is an upper bound. Our first result establish an upper limit on the number of iterations for the duo-item bisection auction.

Proposition 1 *The duo-item bisection auction has an upper limit on the number of iterations of* $6(R-1)$.

Proof For item 1 there can at most be two splits which would generate three prices. The same is true item 2. Without any split $2R$ iterations is required in the duo-item bisection auction. A split at $t^* < R$ starts two new processes and each process requires $R - t^*$ iterations to generate a price and before the split there was $R - t^*$ iterations. Hence in total a split at t^* adds $R - t^*$ number of iterations. The earliest the first split can arise is in iteration 1 and the second split in the next iteration. Thus, the upper limit for the number of iterations is $2(R + (R-1) + (R-2)) = 6(R-1)$, which is reached if six prices are generated and all four splits happens as early as possible.

The VCG outcome is an efficient allocation and the prices are uniquely determined. It is the benchmark for any auction. Loosely speaking the price paid by bidder i assigned item j equals the externality bidder i imposes on the others by its existence in the economy. Given an efficient assignment x the VCG price of item j depends on how the efficient allocation looks in the economy without the bidder who originally was assigned item j.

In specifying the VCG outcome we need to give an assignment and an associated price vector. To define the VCG outcome in a concise manner let $v(i, j) = v_{i1} + v_{j2}$. Now, the VCG outcome is defined as,

$$x_{i'} = 1,\ x_{j'} = 2, \text{ where } (i', j') \in \underset{(i,j)\in N\times N}{\arg\max}\ v(i, j) \text{ and } x_k = 0 \text{ for all other } k \in N. \tag{3}$$

The prices for item 1 and 2 are,

$$\begin{aligned} p_1 &= \max_{(i,j)} v(i, j) - v_{j'2}, \quad \text{where } (i, j) \in (N_{-i'} \times N_{-i'}) \\ p_2 &= \max_{(i,j)} v(i, j) - v_{i'1}, \quad \text{where } (i, j) \in (N_{-j'} \times N_{-j'}). \end{aligned} \tag{4}$$

Computing the VCG outcome in the generic case with two items requires the three highest valuations for both items. To determine an efficient allocation it is sufficient to know the two highest valuations for item 1 and 2, but for the prices we need the three highest valuations. The next result establish that the duo-item bisection auction reaches the VCG outcome. In other words it leads to an efficient assignment and the prices in the allocation are equal to the prices given by Eq. (4) above.

Proposition 2 *The duo-item bisection auction results in the VCG outcome under truthful bidding.*

Proof Under truthful bidding prices generated in the duo-item bisection auction equals the true valuations. In other words generating prices is the same as eliciting preferences. Algorithm 2 elicits the three highest valuations for item 1 and 2. This is the information on preferences required to compute the VCG outcome. Hence, the function v can be restricted to this domain and still the VCG outcome can be computed by Eqs. (3) and (4) above. Furthermore, the function v equals f on this restricted domain and therefore the allocation found by Eq. (1) and (2) must be the same as the VCG outcome.

The plan for the rest of this section is to prove that the modified duo-item bisection auction elicits the minimal amount of information on preferences, required to find the VCG outcome for any sequential elicitation method and all conceivable valuations V.

Lemma 1 *In any sequential elicitation method reaching the VCG outcome for all conceivable valuations V it is necessary to elicit the three highest valuations for item 1.*

Proof Suppose there is only one bidder with the highest valuation for item 1 and 2, then the highest valuation for item 1 is required to determine an efficient allocation. The next example shows that the second and the third highest valuations are necessary to find the VCG outcome. Consider scenario (I) with the efficient assignment $x_{i_{11}} = 1,\ x_{i_{21}} = 2$ and $|T_{1,1} \cup T_{2,1}| = 2$. Suppose bidder i_{21} assigned item 2 also has the second highest valuation for item 1, formally $|T_{1,2} \cup T_{2,1}| = 1$. Similarly bidder i_{11} assigned item 1 has the second highest valuation for item 2, formally $|T_{1,1} \cup T_{2,2}| = 1$. Then there are two possible efficient assignments in the economy $N_{-i_{11}}$, either $x_{i_{12}} = 1$, $x_{i_{23}} = 2$, or $x_{i_{13}} = 1$, $x_{i_{21}} = 2$. The latter is efficient when $v_{i_{12}1} + v_{i_{23}2} \leq v_{i_{13}1} + v_{i_{21}2}$ and if the inequality is reversed the former is efficient.[1] Hence, we need $v_{i_{12}1}$, $v_{i_{13}1}$, $v_{i_{22}2}$ and $v_{i_{21}2}$ to find the VCG price for item 1. Thereby we can conclude that it is necessary to elicit all the three highest valuations for item 1.

Lemma 2 *With truthful bidding the modified duo-item bisection auction elicits a necessary amount of information on preferences to reach the VCG outcome for all conceivable valuations V.*

Proof The proof follows the structure of Algorithm 3. We will show step by step that the valuations elicited in Algorithm 3 are required to find the VCG outcome. First, to find an efficient assignment the information on who has the highest valuation is required. Next, if we are in scenario (II) or (III), the second highest valuation for item 2 is required to find an efficient assignment. For scenario (I) the second highest valuation for item 2 is required to compute the VCG prices. Hence, the information elicited on preferences in step 1 is necessary to find the VCG outcome.

Moving on to step 2 where Algorithm 3 proceeds to when we have scenario (II) or (III). To find the efficient assignment the highest valuation for item 2 is required in both scenarios.

- Let us begin with scenario (II) and see why it is necessary sometimes to elicit the third highest valuation for item 2. The efficient assignment for scenario (II) is to give item 1 to bidder i_{11} and item 2 to bidder i_{22}. One candidate for the VCG price for item 2 equals $v_{i_{23}2}$. Hence, the third highest valuation for item 2 should be elicited.
- Looking at scenario (III), where the efficient assignment is to give item 1 to bidder i_{12} and item 2 to bidder i_{21}. Impose further the condition of $|T_{1,2} \cup T_{2,2}| = 1$. One candidate for the efficient assignment in $N_{-i_{21}}$ is when bidder i_{12} keeps item 1 and bidder i_{23} is assigned item 2. To evaluate this candidate we need to know the third highest valuation for item 2. Thus, it is necessary to elicit the information on preferences as described in step 2 of Algorithm 3.

Next consider step 3 in Algorithm 3, where we end up if $|T_{1,1} \cup T_{2,1}| = 2$. In the efficient assignment bidder i_{11} is given item 1 and bidder i_{21} is given item 2. At step 3 in Algorithm 3 there are two cases to consider (i) and (ii).

- First, consider case (i) with two bidders having the highest valuation for item 2. Then letting bidder i_{11} keep item 1 and giving item 2 to the bidder with the third highest valuation is a candidate for an efficient assignment in the economy $N_{-i_{21}}$. Hence, eliciting the third highest valuation is necessary to find the VCG price for item 2.
- The other case (ii) is when one bidder has the highest valuation for item 2. To complicate the matter there are two possible subcases (a) and (b), when one bidder has the highest valuation for item 2.

[1] If equality both assignments are efficient, and both of them are possible choices.

1. In sub-case (a) of Algorithm 3 at step 3.(ii), where there is one bidder with the highest valuation for item 1. If $|T_{1,2} \cup T_{2,1}| = 1$ the highest valuation for item 2 is elicited. It is required to evaluate the candidate for an efficient assignment in $N_{-i_{11}}$. The candidate consists of giving item 1 to bidder i_{13} and item 2 to bidder i_{21}. Similarly when $|T_{1,1} \cup T_{2,2}| = 1$ the third highest valuation for item 2 is required to evaluate the efficient assignment in $N_{-i_{21}}$.
2. In sub-case (b) of Algorithm 3 at step 3.(ii) the highest valuation for item 2 is always elicited. Since, one candidate for an efficient assignment in the economy $N_{-i_{11}}$ is to give item 1 to bidder i_{12} and item 2 to bidder i_{21}. The third highest valuation is elicited only if $|T_{11} \cup T_{22}| = 2$. Because one candidate for an efficient assignment in $N_{-i_{21}}$ is to give item 1 to bidder i_{21} and item 2 to bidder i_{23}, recall that bidder i_{21} also has the highest valuation for item 1.

Now it remains to look at step 4 in Algorithm 3, where we end up if $|T_{1,1} \cup T_{2,1}| \geq 3$. This is straightforward. The VCG price for item 1 respectively 2 equals the second highest valuation for item 1 respectively 2.

Lemma 3 *With truthful bidding the modified duo-item bisection auction elicits a sufficient amount of information on preferences to reach the VCG outcome for all conceivable valuations V.*

Proof The proof goes through each of the three scenarios (I), (II) and (III). For each of them we establish that the information on preferences suffices to find the VCG outcome. To begin with we can conclude that in all three scenarios. The information on preferences suffices to determine an efficient assignment. Lemma 3 boils down to whether the available information on preferences suffices to determine the VCG prices.

Let's first look at scenario (II) and (III). In scenario (II) Algorithm 3 elicits each of the three highest valuations' for item 2, this always suffices to find the VCG outcome. Since, eliciting the three highest valuations for both items is the maximal amount of information required to determine the VCG outcome with two. In scenario (III) both of the two highest valuations' for item 2 are elicited. The third highest valuation for item 2 is not elicited when $|T_{1,2} \cup T_{2,2}| \geq 2$. This causes no problem because bidder i_{23} is neither an alternative for item 2 in $N_{-i_{12}}$ nor in $N_{-i_{21}}$.

Now, consider the remaining scenario (I). The case with $|T_{1,1} \cup T_{2,1}| \geq 3$ was covered in the proof of Lemma 2 and there is nothing to be added. Moving on to the other case in scenario (I) with $|T_{1,1} \cup T_{2,1}| = 2$. It is enough to discuss the cases when Algorithm 3 does not elicit all of the three highest valuations' for item 2. Which means that it suffices to consider cases in scenario (I) where one bidder has the highest valuation for item 2. Taking all this together it remains to cover cases in scenario (I) with $|T_{1,1} \cup T_{2,1}| = 2$ and $|T_{2,1}| = 1$

- Begin by looking at the case with $|T_{1,1}| = 2$ and $|T_{2,1}| = 1$. In this case we are in step 3.(ii)(a) of Algorithm 3. The highest and the second highest valuation for item 2 are elicited. The third highest valuation is not elicited when $|T_{1,1} \cup T_{2,2}| \geq 3$. This does not cause any problems. The VCG price for item 2 equals $v_{i_{22}2}$ and for item 1 there are two candidates for an efficient assignment in $N_{-i_{11}}$. The first candidate is to give item 1 to bidder i_{11} and item 2 to bidder i_{22}. In the other alternative item 1 is given to bidder i_{12} and item 2 to bidder i_{21}. Hence, the third highest valuation is not required.

– Next consider the case with $|T_{1,1}| = |T_{2,1}| = 1$. Here there are four sub-cases.
  1. When $|T_{1,2} \cup T_{2,1}| \geq 2$ and $|T_{1,1} \cup T_{2,2}| \geq 2$ the VCG price for both items equals the second highest valuation for each item, and we conclude that eliciting the second highest valuation, as is done in Algorithm 3, for item 2 is enough.
  2. Now, suppose one of the conditions changes. Say $|T_{1,2} \cup T_{2,1}| = 1$ and the other condition remains, $|T_{1,1} \cup T_{2,2}| \geq 2$. The VCG price for item 2 is still equal to $v_{i_{22}2}$, but the price for item 1 changes. In Algorithm 3 the highest valuation for item 2 is elicited. This is all what is needed to find the VCG price for item 1. Since, bidder i_{22} has the highest valuation for item 2 amongst remaining bidders in $N_{-i_{11}}$. Hence, no need to elicit the third highest valuation for item 2.
  3. A similar argument can be made when $|T_{1,2} \cup T_{2,1}| \geq 2$ and $|T_{1,1} \cup T_{2,2}| = 1$. The price for item 1 equals $v_{i_{12}1}$ and the VCG price for item 2 can be found with the knowledge of the second and the third highest valuation for item 2. bidder i_{21} cannot be assigned any item in $N_{-i_{21}}$.
  4. Finally when both $|T_{1,2} \cup T_{2,1}| = 1$ and $|T_{1,1} \cup T_{2,2}| = 1$ all three highest valuations for item 2 are elicited.

Thus, in scenario (I) Algorithm 3 elicits the required amount of information on preferences to find the VCG outcome.

Theorem 1 *The modified duo-item bisection auction elicits the minimal amount of information on preferences to reach the VCG outcome for all conceivable valuations V, and for any sequential elicitation method.*

Proof Lemma 1,2 and 3 taken together proves Theorem 1. □

5 Conclusion

We have proposed a multi-item bisection auction, focusing on the case with two items for sale. It is a generalization of the single item auction (Grigorieva et al. 2007). The analysis of the auction has been carried out in the environment with two items for sale. The auction results in the VCG outcome. As part of the analysis in the environment with two items for sale a modified version was presented. We proved that the modified version elicits the minimal amount of information on preferences required to reach the VCG outcome for any sequential elicitation method. In other words the information acquired is sufficient and necessary for computing the VCG outcome.

In this sense it resembles the idea of the single item bisection auction, where only the second highest valuation is revealed. However, with two items for sale it is not enough to elicit only the second highest valuation to compute the VCG outcome. More information on preferences is required. The other agents' valuations are only revealed up to the point such that the winner, the second highest and the third highest valuation can be determined.

An open question, that is not addressed within this paper, is the trade-off between speed of the auction and efficiency. In the current formulation increases are made in unit steps. A method to improve the speed of the auction is to allow for larger increases in prices. Since we do not have a process for decreasing prices, the problem of increasing prices with more than one is that we might obtain inefficient outcomes.

Acknowledgments I would like to thank the anonymous referee for detailed comments and criticism. I am also grateful to participants in the seminar at Lund University, the workshop at the summer school on bargaining theory and applications in Turko, the conference for economic design in Montreal 2011, and the workshop at the Arne Ryde Memorial Lectures on Dynamic Pricing: A Mechanism Design Approach in Lund 2011. I would like thank Jean-Jacques Herings, Bo Larsson and in particular Tommy Andersson for valuable comments.

References

Andersson, T., & Andersson, C. (2012). Properties of the DGS-auction algorithm. *Computational Economics*, *39*, 113–133.

Ausubel, L. M. (2004). An efficient ascending-bid auction for multiple objects. *American Economic Review*, *94*(5), 1452–1475.

Clarke, E. H. (1971). Multipart pricing of public goods. *Public Choice*, *XI*, 17–33.

Demange, G., Gale, D., & Sotomayor, M. (1986). Multi-item auctions. *Journal of Political Economy*, *94*, 863–872.

Engelberecht-Wiggans, R., & Kahn, C. M. (1991). Protecting the winner: Second-price versus oral auctions. *Economics Letters*, *35*(3), 243–248.

Grigorieva, E., Herings, P. J. J., Müller, R., & Vermeulen, D. (2007). The private value single item bisection auction. *Economic Theory*, *30*, 107–118.

Groves, T. (1973). Incentives in teams. *Econometrica*, *61*, 617–631.

Gul, F., & Staccetti, E. (2000). The English auction with differentiated commodities. *Journal of Economic Theory*, *92*, 66–95.

Hudson, B., & Sandholm, T. (2004). Effectiveness of query types and policies for preference elicitation in combinatorial auctions. In *Autonomous agents and multi-agent systems*. Washington, DC: IEEE Computer Society.

Mishra, D., & Parkes, D. (2009). Multi-item Vickrey–Dutch auctions. *Games and Economic Behavior*, *66*, 326–347.

Parkes, D. C. (2001). An iterative generalized Vickrey auction: Strategy-proofness without complete revelation. In *Proceedings of AAAI Spring Symposium on Game Theoretic and Decision Theoretic Agents* (pp. 78–87). AAAI Press.

Perry, M., & Reny, P. J. (2005). An efficient multi-unit ascending auction. *Review of Economic Studies*, *72*, 567–592.

Rothkopf, M. H., Tisberg, T. J., & Kahn, E. P. (1990). Why are Vickrey auctions rare. *Journal of Political Economics*, *98*, 94–109.

Vickrey, W. (1961). Counterspeculation, auctions and competetive sealed tenders. *Journal of Finance*, *16*, 8–37.