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A method for finding the maximal set in excess demand[☆]

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HIGHLIGHTS

- We analyse the set of individual rational payoffs where no agent set is overdemanded.
- We present a polynomial time method for identifying the maximal set in excess demand.
- This method can be used to find the unique minimum element in the set of interest.

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ABSTRACT

We present a polynomial time method for identifying the maximal set in excess demand at a given payoff vector. This set can be used in "large" partnership formation problems to identify the minimum element in the set of individually rational payoff vectors at which there is no overdemanded set of agents. This minimum element corresponds to the minimum Walrasian equilibrium price vector in a special case of the partnership formation problem.

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1. Introduction

Several recent papers have investigated the *partnership formation problem*.[1] This problem involves a set of agents who can stay independent or form a partnership with some other agent if it is in their mutual interest to do so. Agents that stay independent generate a value for themselves whereas a cooperating pair must agree upon how to split their jointly generated value. A prominent special case of this problem is the *assignment game* (Shapley and Shubik, 1971), where the agents are split into two disjoint groups (e.g., buyers and sellers) and the roles of the agents are fixed.

In contrast to the assignment game, equilibrium may fail to exist for a partnership formation problem due to its one-sided nature. A more positive result is due to Andersson et al. (2014) who show that the set of individually rational payoff vectors at which there is no *overdemanded* set contains a unique minimum element $p^{\min}$ that can be used to determine whether an equilibrium exists or not.[2] Their proof is constructive as it is based on an algorithm for identifying $p^{\min}$. Each iteration of the algorithm determines whether there is an overdemanded set or not. If so, a minimal such set is identified.[3] However, for their algorithm to terminate, an exhaustive search through *all* subsets of agents is required which makes the algorithm computationally infeasible for problems involving many agents.

The main innovation of this note is to present a polynomial time method for identifying a maximal set *in excess demand* at a given payoff vector. By construction, this set shares an important property with the minimal overdemanded sets examined by Andersson et al. (2014).[4] It turns out that a maximal set in excess

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1 See Alkan and Tuncay (2013), Andersson et al. (2014), Chiappori et al. (2012), and Talman and Yang (2011).

2 A set of agents S is *overdemanded* at a payoff vector if the number of agents demanding only agents in the set S is strictly greater than the number of agents in the set S.

3 An overdemanded set S is *minimal* if no proper subset of S is overdemanded.

4 The notion of a set in excess demand is weaker than the notion of a minimal overdemanded set (Demange et al., 1986) as demonstrated by van der Laan and

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demand, therefore, can be used as a termination criterion *and* to update payoffs in a modified version of the algorithm in Andersson et al. (2014) without altering its nice convergence properties. Furthermore, there is a unique maximal set in excess demand, making the path – the sequence of payoff vectors traversed from the start payoffs to $p^{\min}$ – unique. This is important as there are typically billions of different paths connecting these two payoff vectors even for "small" problems. An advantage is hence that no additional selection rule is needed for determining the exact path. In addition, we show through simulations that the algorithm typically converges in fewer iterations when using the maximal set in excess demand rather than an arbitrary set in excess demand. In roughly 96% of the investigated instances, the algorithm terminates weakly faster when based on the maximal set. Moreover, it requires on average 16%–21% fewer iterations than if the set is chosen arbitrarily among the sets in excess demand.

Related to this note, Alkan and Tuncay (2013) present a polynomial time algorithm for identifying an equilibrium for the partnership formation problem. Their equilibrium notion is, however, not identical to the one used by Andersson et al. (2014). Alkan and Tuncay (2013) introduce the opportunity for agents to form half-partnerships (i.e., agents are allowed to have two half-partners as an alternative to having one full partner). Additionally, their algorithm generally does not converge to $p^{\min}$. Andersson et al. (2013) and Sankaran (1994) have provided different polynomial time methods for identifying the maximal set in excess demand for assignment games.

This note is organized as follows. Section 2 contains the model and some basic definitions. Section 3 describes the polynomial time method for identifying the maximal set in excess demand. Section 4 presents the results of the simulation study.

2. The model and basic definitions

The finite set of *agents* is denoted as $N = \{1, 2, \ldots, n\}$. Each $i \in N$ can stay independent and generate a value of $v_{ii} = 0$ or form a partnership with some other agent $j \neq i$. In the latter case, agents i and j generate the *joint value* $v_{ji} = v_{ij} \in \mathbb{Z}$. Let v be the symmetric $n \times n$ matrix containing v_{ij} as its (i, j)th entry. The pair (N, v) is called a *problem*.[5] A *matching* $\mu : N \to N$ satisfies $\mu(i) = j$ if and only if $\mu(j) = i$. A *payoff vector* is $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$, and it is said to be *individually rational* if $p_i \geq 0$ for all $i \in N$. Agents are assumed to have quasilinear preferences, i.e., the *demand correspondence* for agent $i \in N$ at payoff vector p is

$$D_i(p) = \{j \in N : v_{ij} - p_j \geq v_{ik} - p_k \text{ for all } k \in N\}.$$

The agents who demand only agents in $S \subseteq N$ at p are $\mathcal{O}(S, p) = \{i \in N : D_i(p) \subseteq S\}$. The set S is *overdemanded* at p if $|\mathcal{O}(S, p)| > |S|$. The set of individually rational payoff vectors at which there are no overdemanded sets is given by:

$$\mathcal{H} = \{p \in \mathbb{R}^n : p_i \geq v_{ii} \text{ for all } i \in N \text{ and } |\mathcal{O}(S, p)| \leq |S| \text{ for all } S \subseteq N\}.$$

A payoff vector $p^{\min} \in \mathcal{H}$ is minimum if $p^{\min} \leq p$ for all $p \in \mathcal{H}$. As shown by Andersson et al. (2014) there exists a unique minimum payoff vector $p^{\min} \in \mathcal{H}$ for each problem (N, v). For the assignment game, this payoff vector corresponds to the unique minimum Walrasian equilibrium price vector (Demange and Gale, 1985).

The agents who demand some agent in $S \subseteq N$ at p are $\mathcal{U}(S, p) = \{i \in N : D_i(p) \cap S \neq \emptyset\}$. A set $S \subseteq N$ is in *excess demand* at p if, for all non-empty $T \subseteq S$, the following condition is satisfied:

$$|\mathcal{U}(T, p) \cap \mathcal{O}(S, p)| > |T|. \quad (1)$$

As demonstrated by Andersson et al. (2013) and Mo et al. (1988), there exists a unique maximal set in excess demand whenever there exists an overdemanded set.

3. A method for identifying the unique maximal set in excess demand

Andersson et al. (2014) demonstrate that $p^{\min}$ can be identified using a simple algorithm where the payoffs in each step are increased for agents in an arbitrary minimal overdemanded set. Importantly, this algorithm still converges to $p^{\min}$ if the payoff increases and the termination criterion instead is based on the maximal set in excess demand, as this set by construction satisfies a condition examined by Andersson et al. (2014, see Lemma 1). As discussed in the Introduction, there are several benefits to making this modification. Of vital importance for larger problems is what is demonstrated next: the maximal set in excess demand can be found in polynomial time.

To identify the set, we use a directed bipartite graph G constructed from an artificial problem $(N \cup N', w)$. Interpret each $i' \in N' = \{1', 2', \ldots, n'\}$ as the "clone" of agent $i \in N$, and let

$$w_{j'i} = w_{ij'} = \begin{cases} v_{ij} & \text{if } i \in N \text{ and } j' \in N' \text{ is the clone of } j \in N \\ -1 & \text{otherwise.} \end{cases}$$

With some abuse of notation, extend p such that $p_{i'} = p_i$ for each clone $i' \in N'$ of $i \in N$. Note that $D_i(p) \subseteq N'$ for all $i \in N$ in $(N \cup N', w)$ as $w_{ii'} = 0 > w_{ij}$ for all $j \in N$. In particular, $j \in D_i(p)$ in (N, v) *if and only if* the corresponding clone $j' \in D_i(p)$ in $(N \cup N', w)$. Hence, a set S is in excess demand in (N, v) *if and only if* the corresponding clones S' are in excess demand in $(N \cup N', w)$.

A *matching that satisfies demand* is $\chi : N \to N' \cup \{\emptyset\}$, where, for all $i \in N$ such that $\chi(i) \neq \emptyset$, $\chi(i) \in D_i(p)$. Let $\chi^{-1}(T') \equiv \{i \in N : \chi(i) \in T'\}$ be the agents matched to the clones in $T' \subseteq N'$. Assume throughout that χ is maximal in the sense that $\chi^{-1}(N') \not\subset \tilde{\chi}^{-1}(N')$ for each matching that satisfies demand $\tilde{\chi}$. Collect the unmatched agents in $U \equiv \{i \in N : \chi(i) = \emptyset\}$.

Next, we construct the graph $G = (V, E)$. The vertex set V is $N \cup N'$. The edge set E contains an arc from $i \in N$ to $j' \in N'$ whenever $j' \in D_i(p)$ and $\chi(i) \neq j'$ and an arc from $j' \in N'$ to $i \in N$ whenever $\chi(i) = j'$. In G, $t \in V$ is *reachable* from $s \in V$ *through* $v_k \in V$ if there exists a sequence $(s = v_0, v_1, \ldots, v_m = t)$ such that v_k is adjacent to v_{k+1} for all $k = 0, 1, \ldots, m - 1$. Let

$$R' \equiv \{j' \in N' : j' \text{ is reachable from some } i \in U\}.$$

Collect their matches in $R \equiv \chi^{-1}(R')$. Note that $D_i(p) \subseteq R'$ for all $i \in U$, hence $U \subseteq \mathcal{O}(R', p)$.

We remark that χ can be found in polynomial time, for instance by using the techniques in Edmonds (1967). In addition, it is possible to check if t is reachable from s in polynomial time (breadth first search or iterative deepening depth-first search algorithms). Consequently, if the method for identifying the maximal set in excess demand is based on the notion of reachable vertices in G, it will have polynomial time complexity. This is the case for the method described in the following theorem. We show that R' is the maximal set in excess demand for $(N \cup N', w)$. Hence, the maximal set in excess demand for (N, v) is exactly the agents whose clones are in R'.

Theorem 1. *Fix a problem (N, v) and a payoff vector p. Construct R' as described above. Then the set of agents whose clones are R' is the maximal set in excess demand at p whenever some set is in excess demand.*

Yang (2008) and Andersson et al. (2013). As also demonstrated by Andersson et al. (2013), the iterative auction algorithm in Demange et al. (1986) always converges to the unique minimum Walrasian equilibrium price vector if it is based on an arbitrary set in excess demand. In addition, the strategic properties of the auction algorithm in Demange et al. (1986) will continue to hold if a set in excess demand in employed instead of a minimal overdemanded set.

[5] A special case of this model is the assignment game (Shapley and Shubik, 1971) where the roles of the agents are given, and all agents in N are exogenously split into two disjoint groups, N_1 and N_2 (with $N_1 \cup N_2 = N$), where agents in the same group cannot be partners. See Talman and Yang (2011).

Table 1
Summary statistics for the simulation study.

Problem size (number of agents)	10	11	12	13	14	15
Average number of iterations MaxED_Path	7.30	9.59	9.07	11.1	10.4	12.2
Average number of iterations Random_Path	8.73	11.2	11.2	13.4	13.2	15.2
MaxED_Path weakly faster than Random_Path (%)	96.5	95.7	95.9	95.8	96.6	96.8
MaxED_Path strictly faster than Random_Path (%)	63.5	66.6	76.1	77.4	83.9	84.5

Proof. If $U = \emptyset$, then $\chi(i) \in D_i(p)$ for all $i \in N$. Then, for all $T' \subseteq N'$, $\mathcal{O}(T', p) \subseteq \chi^{-1}(T')$, as for all $i \notin \chi^{-1}(T')$, $\chi(i) \in D_i(p)$ and $\chi(i) \notin T'$, and hence $i \notin \mathcal{O}(T', p)$. In other words, the agents matched to the clones T', $\chi^{-1}(T')$, are the only ones who can demand clones exclusively in T', as each other agent $i \notin \chi^{-1}(T')$ must demand her match $\chi(i) \notin T'$. Hence, $|\mathcal{O}(T', p)| \leq |\chi^{-1}(T')| = |T'|$. Therefore no set is in excess demand (nor is any set overdemanded).

Assume instead $U \neq \emptyset$. We first demonstrate that $R \subseteq \mathcal{O}(R', p)$. For each $i \in R$, $\chi(i) \in R'$. Hence, i is reachable from some $j \in U$ through $\chi(i)$. But then each $k' \in D_i(p)\backslash\{\chi(i)\}$ is reachable by j through i, and therefore each such $k' \in R$. Hence, $D_i(p) \subseteq R'$ for all $i \in R$.

Next, we show that R' is in excess demand. Take an arbitrary non-empty set $T' \subseteq R'$ and agent $i \in \chi^{-1}(T') \subseteq \chi^{-1}(R') = R$. As just shown, $i \in \mathcal{O}(R', p)$. Additionally, $\chi(i) \in D_i(p) \cap T'$, and hence $i \in \mathcal{U}(T', p)$. By inspecting two cases, we will show that there always exists an agent $m \notin \chi^{-1}(T')$ such that condition (1) is satisfied for T' (and hence R' is in excess demand):

$$|\mathcal{U}(T', p) \cap \mathcal{O}(R', p)| \geq |\chi^{-1}(T') \cup \{m\}| = |T'| + 1 > |T'|.$$

Case 1: There exists $i \in U$ such that $D_i(p) \cap T' \neq \emptyset$. As noted before, $U \subseteq \mathcal{O}(R', p)$. Let $m \equiv i$.

Case 2: For all $i \in U$, $D_i(p) \cap T' = \emptyset$. As T' is reachable from some $i \in U$, there exists $j \in R$ such that $\chi(j) \notin T'$ and $D_j(p) \cap T' \neq \emptyset$. Let $m \equiv j$.

Finally, we demonstrate that R' is the maximal set in excess demand. Suppose, to obtain a contradiction, there exists $S' \nsubseteq R'$ in excess demand. Define $T' \equiv S' \backslash R' \neq \emptyset$. By contradiction, suppose there exists $i \notin \chi^{-1}(T')$ such that $i \in \mathcal{U}(T', p) \cap \mathcal{O}(S', p)$. As $i \in \mathcal{U}(T', p)$ and $T' \cap R' = \emptyset$, $i \notin \mathcal{O}(R', p)$. Therefore, $i \notin U$, and hence $\chi(i) \neq \emptyset$; also, $i \notin R$, so $\chi(i) \notin R'$. As $i \in \mathcal{O}(S', p)$, $\chi(i) \in S'$. But then $\chi(i) \in S' \setminus R' = T'$, a contradiction to $i \notin \chi^{-1}(T')$. Therefore

$$|\mathcal{U}(T', p) \cap \mathcal{O}(S', p)| \leq |\chi^{-1}(T')| = |T'|.$$

Condition (1) is then not satisfied for $T' \subseteq S'$, contradicting S' being in excess demand. □

4. Simulations

To conclude, we report the results of a simulation study based on modifying the algorithm of Andersson et al. (2014). We contrast always increasing payoffs for agents in the maximal set in excess demand (MaxED_Path) with always increasing payoffs for a randomly selected set in excess demand (Random_Path). For each problem size, we examine 1000 instances with uniformly distributed values, and for each instance, we compare MaxED_Path with 1000 random paths. The findings are summarized in Table 1 for problems containing 10–15 agents.

On the top rows of Table 1, the average number of payoff increases needed to converge to $p^{\min}$ is presented for MaxED_Path and Random_Path. The bottom rows contain percentages describing how often MaxED_Path requires weakly and strictly fewer iterations than Random_Path. Notable is that MaxED_Path is weakly faster than between 95.7% and 96.8% of the random paths. It also requires on average 16%–21% fewer iterations to converge to $p^{\min}$.

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